Journal of Algebra 373 (2013) 223–248

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Effectively categorical abelian groups ☆

Rodney Downey [a], Alexander G. Melnikov [b,*]

[a] Department of Mathematics, Statistics, and Operations Research, Victoria University of Wellington, New Zealand
[b] School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore

ARTICLE INFO

Article history:
Received 30 June 2011
Available online 27 October 2012
Communicated by Efim Zelmanov

Keywords:
Abelian groups and modules
Computable model theory
Effective categoricity

ABSTRACT

We study effective categoricity of computable abelian groups of the form $\bigoplus_{i\in\omega} H$, where H is a subgroup of $(Q, +)$. Such groups are called homogeneous completely decomposable. It is well-known that a homogeneous completely decomposable group is computably categorical if and only if its rank is finite.

We study Δ^0_n-categoricity in this class of groups, for $n > 1$. We introduce a new algebraic concept of S-independence which is a generalization of the well-known notion of p-independence. We develop the theory of S-independent sets. We apply these techniques to show that every homogeneous completely decomposable group is Δ^0_3-categorical.

We prove that a homogeneous completely decomposable group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to the free module over the localization of Z by a computably enumerable set of primes P with the semi-low complement (within the set of all primes).

We apply these results and techniques to study the complexity of generating bases of computable free modules over localizations of integers, including the free abelian group.

☆ We are grateful to Andre Nies, Noam Greenberg, and the anonymous referee who suggested numerous corrections to the paper. Many thanks to Iskander Kalimullin for useful and motivative discussions at different stages of the project. We acknowledge support of the Marsden Fund for this project.

* Corresponding author.
E-mail addresses: rod.downey@msor.vuw.ac.nz (R. Downey), amelnikov@ntu.edu.sg (A.G. Melnikov).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.09.020

1. Introduction

1.1. Computable structures and effective categoricity

Remarkably, the study of effective procedures in group theory pre-dates the clarification of what is meant by a computable process; beginning at least with the work of Max Dehn in 1911 [8] who studied word, conjugacy and isomorphisms in finitely presented groups. While the original questions concerned themselves with finitely presented groups, it turned out that they were intrinsically connected with questions about infinite presentations with computable properties. In [22], Graham Higman proved what is now called the Higman Embedding Theorem which stated that a finitely generated group could be embedded into a finitely presented one iff it had a computable presentation (in a certain sense).

The current paper is centered in the line of research of effective procedures in computably presented groups. By computable groups, we mean groups where the domain is computable and the algebraic operation is computable upon that domain.

Such studies can be generalized to other algebraic structures such as fields, rings, vector spaces and the like, a tradition going back to Grete Hermann [21], van der Waerden [44], and explicitly using computability theory, Rabin [40], Mal'tsev [32] and Fröhlich and Shepherdson [17].

More generally, computably presentable algebraic structures are the main objects of study in computable model theory and effective algebra. Recall that for an infinite countable algebraic structure $\mathcal{A}$, a structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is called a *computable presentation of* $\mathcal{A}$ if the domain of $\mathcal{B}$ is (coded by) $\mathbb{N}$, and the atomic diagram of $\mathcal{B}$ is a computable set. If a structure has a computable presentation then it is *computably presentable*. In the same way that *isomorphism* is the canonical classification tool in classical algebra, when we take presentations into account, *computable* isomorphism becomes the main tool. Now two presentations are regarded as the same if they agree up to computable isomorphism. However, an infinite computably presentable structure $\mathcal{A}$ may have many of different computable presentations. Such differing presentations reflect differing computational properties. For example, a computable copy of the order type of the natural numbers might have the successor relation computable (as the familiar presentation does), whereas another might have this successor relation non-computable. Such copies cannot be computably isomorphic.

An infinite countable structure $\mathcal{A}$ is *computably categorical* or *autostable* if every two computable presentations of $\mathcal{A}$ have a computable isomorphism between them. This would mean that the computability-theoretical properties of every copy are identical. Cantor's back-and-forth argument shows that the dense linear ordering without endpoints forms a computably categorical structure. Computable categoricity is one of the central notions of computable model theory (see [15] or [3]). For certain familiar classes of structures we can characterize computable categoricity by algebraic invariants. For instance, a computably presentable Boolean algebra is computably categorical exactly if it has only finitely many atoms [19,29], a computably presentable linear order is computably categorical if and only if it has only finitely many successive pairs [41], and a computably presentable torsion-free abelian group is computably categorical if and only if its rank is finite [20,39].

Computably categorical structures tend to be quite rare, and it is natural to ask the question of how close to being computably categorical a structure is. As mentioned above, we know that a linear ordering of order type $\mathbb{N}$ is not computably categorical since there is the canonical example where the successor relation is computable, and another where the successor relation is not. But if we are given an oracle for the successor relation, then the structure is computably categorical *relative* to that. The halting problem would be enough to decide whether y is the successor of x in such an ordering. This motivates the following definition.

We say that a structure $\mathcal{A}$ is Δ^0_n*-categorical* if every two computable presentations of $\mathcal{A}$ have an isomorphism between them which is computable with oracle $\emptyset^{(n-1)}$, where $\emptyset^{(n-1)}$ is the $(n-1)$-th iteration of the halting problem. Once computably categorical structures in a given class are characterized, it is natural to ask which members of this class are Δ^0_2-categorical. Here the situation becomes more complex. There are only few results in this area, most of them are partial. For instance, McCoy [34] characterizes Δ^0_2-categorical linear orders and Boolean algebras under some extra effectiveness conditions. Also it is known that in general Δ^0_{n+1}-categoricity does not imply Δ^0_n-categoricity in the classes of linear orders [2], Boolean algebras [3], abelian p-groups [5], and ordered abelian groups [36].

Our goal is to give such a higher level classification of effective categoricity for a certain basic class of torsion-free abelian groups.

1.2. Effective categoricity of torsion-free abelian groups

We study Δ^0_2-categorical and Δ^0_3-categorical torsion-free abelian groups. Recall that an abelian group is torsion-free if every nonzero element of this group is of infinite order.

Question. Which computably presentable torsion-free abelian groups are Δ^0_n-categorical, for $n \geqslant 2$?

It is not even clear how to build an example of a Δ^0_{n+1}-categorical but not Δ^0_n-categorical torsion-free abelian group, for each $n > 2$. The main construction of [1] can be used to construct a not Δ^0_n-categorical computable torsion-free group for every n, but results from [1] do not imply this group is Δ^0_{n+1}-categorical. As with the classical theory of torsion-free abelian groups, general questions about isomorphism classes are either extremely difficult or (in a sense described below) impossible. The main difficulty is the absence of satisfactory invariants for computable torsion-free abelian groups which would characterize these groups up to isomorphism. For instance, Downey and Montalbán [14] showed that the isomorphism problem for computable torsion-free abelian groups is Σ^1_1-complete. To say that a problem is Σ^1_1 means that it can be expressed as $\exists f \forall n R(f(n))$ where here the existential quantification is over *functions*, and R is a computable relation. To say that an isomorphism problem is Σ^1_1-complete means that you cannot make the isomorphism problem any simpler, and hence there are no invariants (like dimension) other than the isomorphism type. Therefore, there cannot be a set of invariants which make the complexity of the problem any simpler.

There are better understood subclasses of the torsion-free abelian groups such as the rank one groups, the additive subgroups of the rationals. As we remind the reader in the next section, these groups have a nice structure theory via Baer's theory of *types* (Baer [4]). This theory can be extended to groups that are of the form $\bigoplus_i H_i$ where each H_i has rank 1, a class called the *completely decomposable* groups. As is well-known, Baer's theory extends to this class so we would have some hope of understanding the computable algebra in this setting.

For the present paper, we restrict ourselves to a natural subclass, the *homogeneous completely decomposable* groups which are countable direct powers of a subgroup of the rationals. More formally, we consider the groups of the form $\bigoplus_{i\in\omega} H$, where H is an additive subgroup of $(Q,+)$. These groups in the classical setting were first studied by Baer [4]. The class of homogeneous completely decomposable groups of rank ω is certainly the simplest and most well-understood class of torsion-free abelian groups of infinite rank. Note that, from the computability-theoretic point of view, this is the simplest possible non-trivial case we may consider: every torsion-free abelian group of finite rank is computably categorical. As we will see, even in this classically simplest case the complete answer to the problem does not seem to be straightforward.

To understand the effective categoricity of these groups, we will need both new uses of computability theory in the study of torsion-free abelian groups, and some new algebraic structure theory, as described in the next section.

1.3. A new algebraic notion, and Δ^0_3-categoricity

To study effective categoricity of homogeneous completely decomposable groups, we introduce a new purely algebraic notion of *S-independence*, where S is a *set* of primes. This is a generalization of the well-known notion of p-independence for a single prime p. In the theory of primary abelian groups, p-independence plays an important role. See Chapter VI of [18] for the theory of p-independent sets and p-basic subgroups. We establish several technical facts about S-independent subsets of homogeneous completely decomposable groups. These facts are of independent interest from the purely algebraic point of view. For instance, Theorem 4.10 essentially shows that

S-independence and free modules over a localization of Z play a similar role in the theory of completely decomposable groups as p-independence and p-basic subgroups do in the theory of primary abelian groups.

This paper essentially studies the effective content of S-independence. We will observe that S-independence in general implies linear independence. Effective content of linearly independent sets was studied in the theory of computable vector spaces (see, e.g., [38]). The notion of S-independence seems to be an adequate replacement of linear independence in the case of free modules over a localization of Z (see Lemma 4.4).

We apply the algebraic techniques developed for S-independent sets to establish an upper bound on the complexity of isomorphisms.

Theorem. *Every homogeneous completely decomposable group is Δ^0_3-categorical.*

This result is sharp: there exist homogeneous completely decomposable groups which are not Δ^0_2-categorical so that we cannot replace Δ^0_3 by Δ^0_2. Also, a homogeneous completely decomposable group of rank ω is never computably categorical (folklore). It is natural to ask for a necessary and sufficient condition for a homogeneous completely decomposable group to be Δ^0_2-categorical. Remarkably, there is a natural condition on the group classifying exactly when this happens.

1.4. Free modules, semi-low sets, and Δ^0_2-categoricity

Certain homogeneous completely decomposable groups may be viewed as free modules over localizations of integers by sets of primes. More specifically, let P be a set of primes which is not the set of all primes, and let $Q^{(P)}$ be the additive subgroup of the rationals $(Q,+)$ generated by fractions of the form $\frac{1}{p^m}$, where $p \in P$ and $m \in \omega$. Let G_P be the direct sum of countably many presentations of $Q^{(P)}$: $G_P = \bigoplus_{i\in\omega} Q^{(P)}$. Baer [4] showed that the classical isomorphism of a homogeneous completely decomposable group $\bigoplus_{i\in\omega} H$ is determined by the *characteristic* of H (see Definition 2.4). If the reader is familiar with the concept of *characteristic*, then she or he may observe that a characteristic consisting of only ∞ and 0 correspond to a group of the form G_P. We characterize the case where a computable completely decomposable homogeneous group is Δ^0_2-categorical via a combination of an algebraic (the group must be of the form G_P) and a mild effectiveness consideration (the complement of the corresponding set P is *semi-low*). That is, P must resemble a computable set in the sense that it has a weak guessing procedure for membership, called *semi-lowness*.

We say that a set S is *semi-low* if the set $H_S = \{e\colon W_e \cap S \neq \emptyset\}$ is computable in the halting problem. As the name suggests (for co-c.e. sets) this is weaker than being low (meaning that $A' \equiv_T \emptyset'$, since every low c.e. set is one with a semi-low complement, but not conversely, see Soare [42,43]). Semi-low sets are connected with the ability to give a fastest enumeration of a computably enumerable set as discovered by Soare [42]. In that paper, Soare showed that if **a** is a c.e. degree which is nonlow, then it contains a c.e. set whose complement is not semi-low. Semi-low sets also appear naturally when one studies automorphisms of the lattice $\mathcal{E}$ of computably enumerable sets under the set-theoretical inclusion. Soare (see, e.g., [43, Theorem 1.1 on p. 375]) showed that if a c.e. set S has a semi-low complement, then the lattice of all c.e. sets is isomorphic to the principal filter $\mathcal{L}(S)$ of c.e. supersets of S. Furthermore, if a c.e. set S has a semi-low complement, then $\mathcal{L}(A)/\mathcal{F}$ is effectively isomorphic to $\mathcal{E}/\mathcal{F}$, where $\mathcal{F}$ stands for the ideal of finite sets. There exist variations of semi-lowness which appear naturally in the study of lattice-theoretic properties of c.e. sets. We say that a set S is semi-low$_{1.5}$ if $\{e\colon W_e \cap S$ is finite$\}$ is computable in $\emptyset''$. Maass [31] showed that if A is c.e. and co-infinite, then $\mathcal{L}(A)/\mathcal{F}$ is effectively isomorphic to $\mathcal{E}/\mathcal{F}$ if and only if $\overline{A}$ is semi-low$_{1.5}$. For more information about semi-low and semi-low$_{1.5}$ sets see [43]. We mention that a c.e. degree is low if and only if it contains semi-low$_{1.5}$ co-c.e. set [11].

It is rather interesting that semi-lowness appears in the characterization of Δ^0_2-categorical abelian groups:

Theorem. *A computable homogeneous completely decomposable group A of rank ω is Δ^0_2-categorical if and only if A is isomorphic to G_P, where P is a c.e. set of primes such that $\{p\colon p$ prime and $p \notin P\}$ is semi-low.*

In particular, if P is c.e. and low, then G_P is Δ^0_2-categorical. As far as we know, this is the first application of semi-low sets in effective algebra. Also, the proof of the theorem above is of some technical interest as it splits into several cases depending on the manner by which the type of the group A is enumerated. The flavor of this proof is that of the “limitwise monotonic” proofs in the literature but is a lot more subtle. The method has a number of new ideas which would seem to have other applications.

1.5. *A coding, and further applications*

Note that the map $P \to G_P$ gives an effective coding of a computably enumerable set of primes into a computable abelian group. Furthermore, P defines G_P uniquely up to isomorphism.

Before we pass to the next result, we briefly discuss similar codings of sets into isomorphism types of various classically simple structures. Effective content of such codings has been intensively studied in recent years. In the theory of computable abelian groups, at least two examples of this kind should be mentioned. See [10] for similar examples in the class of linear orders which led to the notions of η-presentable sets and strongly η-presentable sets.

The first example is the coding of a given set of primes S into the abelian group $G(S) = \bigoplus_{p\in S} Q^{(\{p\})}$, where $Q^{(\{p\})}$ was defined above. Khisamiev [25] showed that $G(S)$ has a computable representation with a certain strong basis exactly if the set S belongs to a certain proper subclass of non-hh-immune Σ^0_2-sets. Khisamiev also asked for a necessary and sufficient condition for the group $G(S)$ to have a computable (decidable) presentation. Downey, Goncharov, Knight et al. [12] showed that $G(S)$ has a computable (decidable) presentation if and only if S is Σ^0_3 (Σ^0_2). Although the group is classically simple, the proof is not straightforward.

The second example of this kind is the coding of a given set of natural numbers S into the abelian p-group which is the direct sum of cyclic groups of orders p^s, one component for each s. Khisamiev [24] showed that this group has a computable presentation if and only if the set S has an effective monotonic approximation from below. Such sets are often called limitwise monotonic [26]. Khisamiev built an example of a Δ^0_2 set which has no such a monotonic approximation ([24]; see [26] for an alternate proof). Limitwise monotonic sets have applications in other fields of computable model theory [26,23,6], and have connections to degree theory [13]. This example also illustrates that the arithmetical complexity does not always reflect the needed effective properties of abelian groups.

We observe that the following are equivalent: (1) G_P is computably presentable, (2) G_P is computably presentable as a module over $Q^{(P)}$ (to be specified), (3) the set of primes P is computably enumerable. See Proposition 3.6 for the proof. Nonetheless, the complexity of a c.e. set P is reflected in G_P via the complexities of possible isomorphisms between computable presentations of G_P. Let $p_0, p_1, \dots$ be the standard listing of primes. As a consequence of the main results of the paper, we obtain:

Theorem. *A co-c.e. set S is semi-low if and only if the group G_P is Δ^0_2-categorical, where $P = \{p_i\colon i \notin S\}$.*

This gives a characterization of semi-low co-c.e. sets in terms of effective algebra. Using the techniques of the paper one can easily show that the weak jump $H_{\widehat{P}}$ of the complement $\widehat{P}$ of P (within the set of all primes) computes some isomorphism between any two computable copies of G_P. It is also not hard to show that $H_{\widehat{P}}$ is indeed the *degree of categoricity* of G_P, for every c.e. set of primes P (see [16] for the definition and for more about degrees of categoricity). Although we do not develop this subject any further, we note that this is the first natural example of an algebraic structure having the weak jump of an encoded set as its degree of categoricity. It also follows from our observation and well-known facts about semi-low sets (see, e.g., [43, pp. 72–73]) that a c.e. degree is high if and only if it contains a c.e. set of primes P such that the group G_P has two computable copies with an isomorphism between these copies which computes $0''$. This shows we cannot improve the upper bound on the complexity of isomorphisms: every homogeneous completely decomposable group is Δ^0_3-categorical, and this is the best we can get even for the groups of the form G_P.

We also apply the main results of the paper to study the complexity of the bases of G_P which generate it as a free module over $Q^{(P)}$. We will see that effective categoricity of G_P can be equivalently reformulated in terms of bases. Our interest is also motivated by the recent results on computable free *non-abelian* groups. More specifically, the computational complexities of sets of generators in free *non-abelian* groups were studied in [7] and [30]. We show:

Theorem. *If a computable presentation of G_P has a Σ^0_2-basis which generates it as a free $Q^{(P)}$-module, then this presentation possesses a Π^0_1-basis which generates it as a free $Q^{(P)}$-module.*

As a consequence of this theorem and the main results of the paper, if $\{p\colon\ p \text{ prime and } p \notin P\}$ is semi-low, then G_P has a Π^0_1-basis which generates it as a free $Q^{(P)}$-module. Thus, every computable copy of the free abelian group has a Π^0_1-basis of generators. This is sharp (folklore).

1.6. The structure of the paper

First, we give some background on the general theory of computable torsion-free abelian groups. Then we develop a bit of the algebraic theory of S-independent sets. Next, we apply this theory to study effective categoricity of homogeneous completely decomposable groups. We conclude the paper stating open problems.

2. Algebraic preliminaries

We use known definitions and facts from computability theory and the theory of abelian groups. Standard references are [43] for computability and [18] for the theory of torsion-free abelian groups. We will see that for our purposes we don't need to use a more complicated two-sorted signature of modules (Proposition 3.6). However, we will use a notation that substitutes the module multiplication (Notation 2.10). Basics of module theory can be found in any classical book on general algebra (see, e.g., [28]).

Definition 2.1 *(Linear independence and rank).* Elements $g_0, \dots, g_n$ of a torsion-free abelian group G are *linearly independent* if, for all $c_0, \dots, c_n \in Z$, the equality $c_0 g_0 + c_1 g_1 + \dots + c_n g_n = 0$ implies that $c_0 = c_1 = \dots = c_n = 0$. An infinite set is *linearly independent* if every finite subset of this set is linearly independent. A maximal linearly independent set is a *basis*. All bases of G have the same cardinality. This cardinality is called the *rank* of G.

We write $A \leqq B$ to denote that A is a subgroup of B. It is not hard to see that a torsion-free abelian group A has rank 1 if and only if $A \leqq \langle Q, +\rangle$.

Definition 2.2 *(Direct sum).* An abelian group G is the *direct sum* of groups A_i, $i \in I$, written $G = \bigoplus_{i\in I} A_i$, if G can be presented as follows:

(1) The domain consists of infinite sequences $(a_0, a_1, a_2, \dots, a_i, \dots)$, each $a_i \in A_i$, such that the set $\{i\colon\ a_i \neq 0\}$ is finite.
(2) The operation $+$ is defined component-wise.

The groups A_i are the *direct summands* or *direct components* of G (with respect to the given decomposition). Note that there may be lots of different ways to decompose the given subgroup. One can check that $G \cong \bigoplus_{i\in I} A_i$, where $A_i \leqq G$, if and only if (1) $G = \sum_{i\in I} A_i$, i.e. $\{A_i\colon\ i \in I\}$ generates G, and (2) for all j we have $A_j \cap \sum_{i\in I,\, i\neq j} A_i = \{0\}$.

We write $k|g$ *in* G (or simply $k|g$ if it is clear from the context which group is considered) and say that k *divides* g *in* G if there exists an element $h \in G$ for which $kh = g$, and we say that h is a k-*root* of g. Note that $k|g$ is simply an abbreviation for the formula $(\exists h)(\underbrace{h + h + \dots + h}_{k \text{ times}} = g)$ in the signature of abelian groups.

If the group G is torsion-free then every $g \in G$ has at most one k-root, for every $k \neq 0$. Assume there were two distinct k-roots, h_1 and h_2, of an element g. Then $k(h_1 - h_2) = 0$ would imply $h_1 = h_2$, a contradiction.

Definition 2.3 *(Pure subgroups and* $[X]$*).* Let G be a torsion-free abelian group. A subgroup A of G is called *pure* if for every $a \in A$ and every n, $n|a$ in G implies $n|a$ in A. For any subset X of G we denote by $[X]$ the least pure subgroup of G that contains X.

For instance, every direct summand of a given group G is pure in G, while the converse is not necessarily the case.

Let us fix the canonical listing of the prime numbers:

$$p_0, p_1, \ldots, p_n, \ldots.$$

Definition 2.4 *(Characteristic and* h_i*).* Suppose G is a torsion-free abelian group. For $g \in G$, $g \neq 0$, and a prime number p_i, set

$$h_i(g) = \begin{cases} \max\{k\colon\ p_i^k | g \text{ in } G\}, & \text{if this maximum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

The sequence $\chi_G(g) = (h_0(g), h_1(g), \ldots)$ is called the *characteristic* of the element g in G.

Thus, for a torsion-free groups G, a subgroup H of G is a pure subgroup of G if and only if $\chi_H(h) = \chi_G(h)$ for every $h \in H$.

Definition 2.5. Let $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$ be two characteristics. Then we write $\alpha \leqslant \beta$ if $k_i \leqslant l_i$ for all i, where ∞ is greater than any natural number.

Definition 2.6 *(Type).* Two characteristics, $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$, are *equivalent*, written $\alpha \simeq \beta$, if $k_n \neq l_n$ only for finitely many n, and k_n and l_n are finite for these n. The equivalence classes of this relation are called *types*.

We write $\mathbf{t}(g)$ for the type of an element g. If $G \leqslant \langle Q, +\rangle$ (equivalently, if G has rank 1) then all nonzero elements of G have equivalent types, by the definition of rank. Hence, we can correctly define the type of G to be $\mathbf{t}(g)$ for a nonzero $g \in G$, and denote it by $\mathbf{t}(G)$. The following theorem classifies torsion-free abelian groups of rank 1:

Theorem 2.7. *(See Baer [4].) Let G and H be torsion-free abelian groups of rank* 1*. Then G and H are isomorphic if and only if* $\mathbf{t}(G) = \mathbf{t}(H)$*.*

The next simplest class of torsion-free abelian groups is the class of *homogeneous completely decomposable* groups.

Definition 2.8 *(Completely decomposable group).* A torsion-free abelian group is called *completely decomposable* if G is a direct sum of groups each having rank 1. A completely decomposable group is *homogeneous* if all its elementary summands are isomorphic.

It is known that any two decompositions of a completely decomposable group into direct summands of rank 1 are isomorphic. Also, two homogeneous completely decomposable groups of the same rank are isomorphic if and only if these groups have the same type [4]. We will refer to this fact by citing Theorem 2.7 since it is a straightforward consequence of this theorem [18]. For instance, a set of primes P defines the group G_P uniquely up to isomorphism.

Definition 2.9. Suppose G is a torsion-free abelian group, g is an element of G, and $n|g$ some n. If $r = \frac{m}{n}$ then we denote by rg the (unique) element mh such that $nh = g$.

Notation 2.10. Let G be an abelian group and $A \subseteq G$. Suppose $\{r_a\colon a \in A\}$ is a set of (rational) indices. If we write $\sum_{a\in A} r_a a$ then we assume that $r_a a \neq 0$ for at most finitely many $a \in A$, and every element $r_a a$ is well-defined in G, according to Definition 2.9. We will use this convention without explicit reference to it.

Now suppose $R \leqq \langle Q, +\rangle$, and $A \subseteq G$. We denote by $(A)_R$ the subgroup of G (if this subgroup exists) generated by $A \subset G$ over $R \leqq Q$, i.e. $(A)_R = \{\sum_{a\in A} r_a a\colon r_a \in R\}$.

Finally, for $R \leqq Q$ and $a \in G$, we denote by Ra the subgroup $(\{a\})_R$ of G.

Let $R \leqq Q$. If a set $A \leqq G$ is linearly independent then every element of $(A)_R$ has the unique presentation $\sum_{a\in A} r_a a$. Otherwise we would have $\sum_{a\in A} r_a a = 0$ for some set of rational indices $\{r_a\colon a \in A\}$, and thus $m\sum_{a\in A} r_a a = \sum_{a\in A} m r_a a = 0$, for some integer m such that $mr_a \in Z$ for all $a \in A$, contrary to our hypothesis. Therefore, $(A)_R = \bigoplus_{a\in A} Ra$ for every linearly independent set A.

3. Computable abelian groups and modules

The notion of a c.e. characteristic is one of the central notions of computable abelian group theory.

Definition 3.1. Let $\alpha = (h_i)_{i\in\omega}$, where $h_i \in \omega \cup \{\infty\}$ for each i, be a characteristic. We say that α is c.e. if the set $\{\langle i, j\rangle\colon j \leqslant h_i,\ h_i > 0\}$ is c.e. (see [37]). This is the same as saying that there is a non-decreasing uniform computable approximation $h_{i,s}$ such that $h_i = \sup_s h_{i,s}$, for every i. Observe that this is a type-invariant property. Thus, a type $\mathbf{f}$ is c.e. if α is c.e., for every α in $\mathbf{f}$ (equivalently, for some α in $\mathbf{f}$).

Theorem 3.2 below was rediscovered several times by various mathematicians including Knight, Downey, and others (see, e.g., [9]).

Theorem 3.2. *(See Mal'tsev [33].) Let G be a torsion-free abelian group of rank* 1*. Then the following are equivalent*:

(1) *The group G has a computable presentation.*
(2) *The type $\mathbf{t}(G)$ is c.e.*
(3) *The group G is isomorphic to a c.e. additive subgroup R of a computable presentation of the rationals $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$.*

Furthermore, each c.e. type corresponds to some computably presented subgroup of the rationals. See [37] for a proof. If a group G is homogeneous completely decomposable then $\mathbf{t}(G)$ is also well-defined. The (1) $\leftrightarrow$ (2) part of Theorem 3.2 can be easily generalized to the class of homogeneous completely decomposable groups:

Proposition 3.3. *A homogeneous completely decomposable group G has a computable presentation if and only if $\mathbf{t}(G)$ is c.e.*

See [37] for more details.

Definition 3.4. We say that C is a computable presentation of a module M over a ring R if

(1) the ring R is isomorphic to a c.e. subring R_1 of a computable ring R_2,
(2) C is a computable presentation of M as an abelian group, and
(3) there is a computable function $f : R_1 \times C \to C$ which maps (r, m) to $r \cdot m \in C$, for every $m \in C$ and $r \in R_1$.

Recall that $Q^{(P)}$ is the subgroup of the rationals $(Q,+)$ generated by the set of fractions $\{\frac{1}{p^k}\colon k \in \omega \text{ and } p \in P\}$.

Remark 3.5. According to Definition 2.9, for every $r = \frac{m}{n} \in Q^{(P)}$ and a an element of the *group* G_P, the element $ra \in G_P$ is definable by the formula $\Phi_r(x,a) \leftrightharpoons mx = na$ in the language of abelian groups (recall that mx and na are abbreviations).

Proposition 3.6. *The following are equivalent*:

(1) *P is c.e.*
(2) *$Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q,+,\times)$.*
(3) *G_P is computably presentable as an abelian group.*
(4) *G_P is computably presentable as a module over $Q^{(P)}$.*

Proof. The implications (1) → (2) and (2) → (3) are obvious.

(3) → (4). By Proposition 3.3, the characteristic α of G_P is c.e. By Theorem 3.2, $Q^{(P)}$ is isomorphic to a c.e. additive subgroup A of $(Q,+,\times)$. Observe that $Q^{(P)}$ may be considered as a c.e. *subring* of Q, because we can ensure that $1 \in A$. It remains to observe that for each element $g \in G_P$ and each rational $r \in Q_P$, the element rg can be found effectively and uniformly.

(4) → (1). Pick an element g of G_P which is divisible by a prime p if and only if $p \in P$. Thus, $p \in P$ if and only if $(\exists x \in G_P)\, px = g$, proving that P is c.e. □

Remark 3.7. Actually we have shown that *every computable presentation of G_P is already a computable presentation of G_P as a module over $Q^{(P)}$.*

Lemma 3.8. *For a c.e. set of primes P, the following are equivalent*:

(1) *Every computable presentation of the group G_P has a Σ^0_n-basis which generates this presentation as a module over $Q^{(P)}$.*
(2) *The group G_P is Δ^0_n-categorical.*
(3) *The $Q^{(P)}$-module G_P is Δ^0_n-categorical.*

Proof. By Proposition 3.6, the *ring* $Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q,+,\times)$.

(1) → (2). Let A and B be computable presentations of the group G_P. Both A and B have Σ^0_n-bases which generate these groups over $Q^{(P)}$. We map these bases one into another using $0^{(n-1)}$. By Remark 3.5, we can extend this map to an isomorphism effectively, using the c.e. subring $Q^{(P)}$ of Q.

(2) → (3). Observe that every computable *group-isomorphism* between two computable module-presentations of G_P is already a computable *module-isomorphism*.

(3) → (1). Pick a computable presentation H of G_P such that the basis which generates H over $Q^{(P)}$ is computable. If G_P is Δ^0_n-categorical then every computable presentation of G_P has a Σ^0_n-basis which is the image of the computable one in H. □

Thus, from the computability-theoretic point of view, G_P may be alternatively considered as an abelian group or a $Q^{(P)}$-module.

4. *S*-independence and excellent *S*-bases

The notion of p-independence (for a single prime p) is a fundamental concept in abelian group theory (see [18, Chapter VI]). We introduce a certain generalization of p-independence to *sets* of primes:

Definition 4.1 *(S-independence and excellent bases).* Let S be a set of primes, and let G be a torsion-free abelian group. If $S \neq \emptyset$, then we say that elements $b_1, \ldots, b_k$ of G are S-independent in G if

$p|\sum_{i\in\{1,\ldots,k\}} m_i b_i$ in G implies $\bigwedge_{i\in\{1,\ldots,k\}} p|m_i$, for all integers $m_1, \ldots, m_k$ and $p \in S$. If $S = \emptyset$, then we say that elements are S-independent if they are simply linearly independent.

Every maximal S-independent subset of G is said to be an *S-basis* of G. We say that an S-basis is *excellent* if it is a maximal linearly independent subset of G.

It is easy to check that S-independence in general implies linear independence. However, an S-basis does not have to be excellent. Lemma 35.1 in [18] implies that *the free abelian group of rank ω contains a $\{p\}$-basis which is not excellent*.

The main reason why we introduce the notion of S-independence is reflected in the example and the lemma below.

Example 4.2. Let Z^2 be the free abelian group of rank 2, and let e_1 and e_2 be such that $Z^2 = Ze_1 \oplus Ze_2$. Suppose that we need to *test*, given a pair of elements g_1 and g_2, if $Zg_1 + Zg_2 = Z^2$. That is, we wish to be able to say "no" if g_1 and g_2 do not generate Z^2. If g_1 and g_2 together generate the group, then $\{g_1, g_2\}$ should be linearly independent. But this is not sufficient: suppose that $g_1 = 2e_0 + e_1$ and $g_2 = e_1$; then $2|g_1 - g_2$, but the element $h = \frac{g_1 - g_2}{2}$ is not in the span of $\{g_1, g_2\}$.

Now we make each Z-component of Z^2 infinitely divisible by 2 and consider the group $Q^{(2)}e_1 \oplus Q^{(2)}e_2$. Note that $2|g_1 - g_2$ in $Q^{(2)}e_1 \oplus Q^{(2)}e_2$, but it is not a problem: it is easy to check that $\{g_1, g_2\}$ generates $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$. In contrast, the elements $h_1 = 3e_0 + e_1$ and $h_2 = e_1$ fail to generate $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$.

More generally, in $Q^{(P)}e_1 \oplus Q^{(P)}e_2$, *the existence of p-roots for $p \in P$ cannot be used to test if two given elements generate the whole group over $Q^{(P)}$ or not.*

Notation 4.3. In this section P stands for a set of primes and $\widehat{P}$ for the complement of P within the set of *all* primes:

$$\widehat{P} = \{p\colon\ p \text{ is prime and } p \notin P\}.$$

Lemma 4.4. *Suppose $G \cong \bigoplus_{i\in I} Q^{(P)}$, and let $B \subseteq G$. Then B is an excellent $\widehat{P}$-basis of G if and only if B generates G as a free module over $Q^{(P)}$.*

Let $\mathcal{P}$ be the set of *all* primes. Then $\widehat{\mathcal{P}} = \emptyset$. Recall that $\emptyset$-independence is simply linear independence, and $G_{\mathcal{P}} \cong D(\omega) = \bigoplus_{i\in\omega} Q$. It is well-known that every maximal linearly independent set generates the vector space $D(\omega)$ over Q. If $P = \emptyset$ then $G_\emptyset \cong FA(\omega) = \bigoplus_{i\in\omega} Z$ is the free abelian group of the rank ω. As a consequence of the lemma, every excellent $\mathcal{P}$-basis of $FA(\omega)$ generates $FA(\omega)$ as a free abelian group.

Proof of Lemma 4.4. ($\Rightarrow$). Let B be an excellent $\widehat{P}$-basis of G. Suppose $g \in G$. By our assumption, B is a basis of G. Therefore, there exist integers m and m_b, $b \in B$, such that $mg = \sum_b m_b b$. Suppose $m = pm'$ for some $p \in \widehat{P}$. By Definition 4.1, $p|m_b$ for all $b \in B$. Therefore, without loss of generality, we can assume that $(m, p) = 1$, for every $p \in \widehat{P}$. By the definition of G, we have

$$g = \sum_b \frac{m_b}{m} b \in (B)_{Q^{(P)}} \leqq G.$$

The set B is linearly independent, therefore $(B)_{Q^{(P)}} = \bigoplus_{b\in B} Q^{(P)}b$ (see the discussion after Notation 2.10). We have $g \in (B)_{Q^{(P)}} \leqq G$ for every $g \in G$. Thus, $G = (B)_{Q^{(P)}}$.

($\Leftarrow$). Let $G = \bigoplus_{b\in B} Q^{(P)}b$ for $B \subseteq G$, and $ph = \sum_{b\in B} m_b b$, where m_b is integer for every $b \in B$, and $p \in \widehat{P}$. We have $h \in G_P$ and thus $h = \sum_{b\in B} h_b$, where $h_b \in Q^{(P)}b$ for each $b \in B$ (recall that $h_b = 0$ for a.e. b).

Therefore $ph = p\sum_{b\in B} h_b = \sum_{b\in B} ph_b = \sum_b m_b b$, and $ph_b = m_b b$ for every b (by the uniqueness of the decomposition of an element). Each direct component of G in the considered decomposition has the form $Q^{(P)}b$. In other words, the element b plays the role of $\mathbf{1}$ in the corresponding $Q^{(P)}$-component of this decomposition. Now recall that $p \notin P$. Thus, $m_b \neq 0$ implies $p|m_b$ for every b, by the definition of $Q^{(P)}$. $\square$

In later proofs we will have to approximate an *excellent* basis stage-by-stage, using a certain oracle. Recall that not every maximal $\widehat{P}$-independent set is an excellent basis of G_P. Therefore, we need to show that, for a given finite $\widehat{P}$-independent subset B of G_P and an element $g \in G_P$, there exists a finite extension $B^\star$ of B such that $B^\star$ is $\widehat{P}$-independent and the element g is contained in the $Q^{(P)}$-span of $B^\star$.

Proposition 4.5. *Suppose $B \subset G_P$ is a finite $\widehat{P}$-independent subset of G_P. For every $g \in G_P$ there exists a finite $\widehat{P}$-independent set $B^\star \subset G_P$ such that $B \subseteq B^\star$ and $g \in (B^\star)_{Q^{(P)}}$.*

Proof. Pick $\{e_i\colon i \in \omega\} \subseteq G_P$ such that $G_P = \bigoplus_{i\in\omega} Q^{(P)}e_i$. Let $\{e_0, e_1, \ldots, e_n\}$ be such that both $B = \{b_0, \ldots, b_k\}$ and g are contained in $(\{e_0, e_1, \ldots, e_n\})_{Q^{(P)}}$. We may assume $k < n$.

Lemma 4.6. *Suppose $B = \{b_0, \ldots, b_k\} \subseteq \bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i$, is a linearly independent set. There exists a set $C = \{c_0, \ldots, c_n\} \subseteq \bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i$, and coefficients $r_0, \ldots, r_k \in Q^{(P)}$ such that*

(1) $\bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i = \bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}c_i$, *and*
(2) $(\{r_0c_0, \ldots, r_kc_k\})_{Q^{(P)}} = (B)_{Q^{(P)}}$.

Proof. It is a special case of a well-known fact [28, Theorem 7.8] which holds in general for every finitely generated module over a principal ideal domain (note that $Q^{(P)}$ is a principal ideal domain). $\square$

We show that if B is $\widehat{P}$-independent (not merely linearly independent) then we can set $B^\star = \{b_0, \ldots, b_k\} \cup \{c_{k+1}, \ldots, c_n\}$, where $C = \{c_0, \ldots, c_n\}$ is the set from Lemma 4.6. Suppose $p|\sum_{0\leqslant i\leqslant k} n_ib_i + \sum_{k+1\leqslant i\leqslant n} n_ic_i$ for a prime $p \in \widehat{P}$. We have

$$\bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i = \bigoplus_{1\leqslant i\leqslant k} Q^{(P)}c_i \oplus \bigoplus_{k+1\leqslant i\leqslant n} Q^{(P)}c_i,$$

and $\sum_{1\leqslant i\leqslant k} n_ib_i \in \bigoplus_{1\leqslant i\leqslant k} Q^{(P)}c_i$. By the purity of direct components, we have $p|\sum_{1\leqslant i\leqslant k} n_ib_i$ within $\bigoplus_{1\leqslant i\leqslant k} Q^{(P)}c_i$ and $p|\sum_{k+1\leqslant i\leqslant n} n_ic_i$ within $\bigoplus_{k+1\leqslant i\leqslant n} Q^{(P)}c_i$. But the former implies $p|n_i$ for all $1 \leqslant i \leqslant k$ by our assumption, and the latter implies $p|n_i$ for all $k+1 \leqslant i \leqslant n$ by the choice of C and Lemma 4.4.

The set $B^\star$ is actually an excellent $\widehat{P}$-basis of $\bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i$, since the cardinality of $B^\star$ is $n+1 = rk(\bigoplus_{i\in\{0,\ldots,n\}} Q^{(P)}e_i)$. Therefore, the set $B^\star = \{b_0, \ldots, b_k\} \cup \{c_{k+1}, \ldots, c_n\}$ is a $\widehat{P}$-independent set with the needed properties. $\square$

Suppose G is a torsion-free abelian group, and $a, b \in G$. Recall that $\chi(a) \leqslant \chi(b)$ iff $h_i(a) \leqslant h_i(b)$ for all i. In other words, $p^k|a$ implies $p^k|b$, for all $k \in \omega$ and every prime p.

Definition 4.7. Let G be a torsion-free abelian group. For a given characteristic α, let $G[\alpha] = \{g \in G\colon \alpha \leqslant \chi(g)\}$.

We have $h_i(a) = h_i(-a)$ and $\inf(h_i(a), h_i(b)) \leqslant h_i(a+b)$, for all i. Furthermore, $\chi(0) \geqslant \alpha$, for every characteristic α. Therefore, $G[\alpha]$ is a subgroup of G.

Definition 4.8. Let $\alpha = (\alpha_0, \alpha_2, \ldots)$. Then $Q(\alpha)$ is the subgroup of $(Q, +)$ generated by elements of the form $1/p_k^x$ where $x \leqslant \alpha_k$.

Example 4.9. Let $\alpha = (\infty, 1, \infty, 1, \ldots, \alpha_{2k} = \infty, \alpha_{2k+1} = 1, \ldots)$. Consider

$$\beta = \alpha + (0, 1, 0, -1, 0, 0, 0, 0, 0, \ldots, 0, \ldots).$$

By Definition 2.6, $\beta \cong \alpha$. Consider the group $H = Q(\alpha)$. We have $1 \in Q(\alpha)$ and $\chi(1) = \alpha$ within $Q(\alpha)$. Note that the characteristic of $a = 3/7$ in $H(\alpha)$ is β. Observe that a/p_{2k}^j belongs to $H[\beta]$, for every $k, j \in \omega$. In contrast, a/p_{2k+1} does not belong to $H[\beta]$. Indeed, the characteristic $\chi_H(a/13)$ is

$$(\infty, 2, \infty, 0, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \ldots)$$

and

$$(\infty, 2, \infty, 0, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \ldots) \ngeq \beta = (\infty, 2, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \ldots).$$

Recall that the type is an equivalence class of characteristics. Thus, the type of $H \leqq Q$ is simply the type of any nonzero element of H. We are ready to state and prove the main result of this section.

Theorem 4.10. *Let $\mathcal{G} = \bigoplus_{i \in \omega} H$, where $H \leqq Q$, $\mathbf{t}(H) = \mathbf{f}$ and $\alpha = (\alpha_0, \alpha_1, \ldots)$ is of type $\mathbf{f}$. Then $\mathcal{G}[\alpha] \cong G_P$, where $P = \{p_i\colon h_i = \infty \text{ in } \alpha\}$. Furthermore, if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G}$ is generated by B over $Q(\alpha)$.*

Informally, this theorem says that each homogeneous completely decomposable group of rank ω has a subgroup isomorphic to G_P, for some P. Furthermore, every excellent $\widehat{P}$-basis of this subgroup generates the whole group G over a certain rational subgroup $Q(\alpha)$ taken as a domain of coefficients. The group $Q(\alpha)$ is not necessarily a ring (recall Notation 2.10). The idea of the technical proof below was essentially illustrated in Example 4.9.

Proof of Theorem 4.10. We prove that $\mathcal{G}[\alpha] \cong G_P$.

Let g_i be the element of the i-th presentation of H in the decomposition $\mathcal{G} = \bigoplus_{i \in \omega} H$ such that $\chi(g_i) = \alpha$. The collection $\{g_i\colon i \in \omega\}$ is a basis of $\mathcal{G}$. Therefore, $\{g_i\colon i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$. By the definition of P, $(\{g_i\colon i \in \omega\})_{Q^{(P)}}$ is a subgroup of $\mathcal{G}[\alpha]$. Furthermore, since $\{g_i\colon i \in \omega\}$ is linearly independent,

$$\left(\{g_i\colon i \in \omega\}\right)_{Q^{(P)}} \cong \bigoplus_{i \in \omega} Q^{(P)} g_i.$$

Thus, we have

$$\bigoplus_{i \in \omega} Q^{(P)} g_i \subseteq \mathcal{G}[\alpha].$$

We are going to show that every element $g \in \mathcal{G}[\alpha]$ is generated by $\{g_i\colon i \in \omega\}$ over $Q^{(P)}$. This will imply $\mathcal{G}[\alpha] \cong G_P$.

Pick any nonzero $g \in \mathcal{G}[\alpha]$. The set $\{g_i\colon i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$, therefore $ng = \sum_{i \in \omega} m_i g_i$ for some integers n and m_i, $i \in \omega$. Since direct components are pure, $n | \sum_{i \in I} m_i g_i$ implies $n | m_i g_i$ for every $i \in \omega$, and $g = \sum_{i \in I} \frac{m_i}{n} g_i$. After reductions we have $g = \sum_{i \in I} \frac{m'_i}{n_i} g_i$, where $\frac{m'_i}{n_i}$ is irreducible. It suffices to show that $\frac{m'_i}{n_i} \in Q^{(P)}$.

Assume there is i such that $\frac{m'_i}{n_i} \notin Q^{(P)}$. Equivalently, for some $p_k \in \widehat{P}$, we have $m'_i \neq 0$ and $n_i = p_k n'_i$, where n'_i is an integer (recall that $\frac{m'_i}{n_i}$ is irreducible).

We have $h_k(\frac{m'_i}{n_i} g_i) = h_k(\frac{m'_i}{n'_i} \frac{g_i}{p_k}) \leqslant h_k(\frac{g_i}{p_k})$, since m'_i is not divisible by p_k. But $h_k(\frac{g_i}{p_k}) < h_k(g_i)$ (recall that $h_k(g_i)$ is finite). It is straightforward from the definitions of h_k that $h_k(g) = \min\{h_k(\frac{m'_i}{n_i} g_i)$: $i \in I,\ m_i \neq 0\}$, since each g_i belongs to a separate direct component of $\mathcal{G}$. Therefore $h_k(g) \leqslant h_k(\frac{m'_i}{n_i} g_i) < h_k(g_i)$. But $\chi(g_i) = \alpha$. Thus, $\chi(g) \ngeqslant \alpha$ and $g \notin \mathcal{G}[\alpha]$, and this contradicts our choice of g. Therefore, $\mathcal{G}[\alpha] \cong G_P$.

We show that if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G} = (B)_{Q(\alpha)}$ (recall Notation 2.10).

For every $b \in B$ consider the minimal pure subgroup which contains b (recall that we denote this group by $[b]$, see Definition 2.3). Consider $\langle B \rangle = \sum_{b \in B} [b] \leqq \mathcal{G}$. In fact $\langle B \rangle = \bigoplus_{b \in B} [b]$, because B is linearly independent within $\mathcal{G}[\alpha]$ and, therefore, within $\mathcal{G}$ as well.

By our choice, $b \in \mathcal{G}[\alpha]$. Thus, $\chi(b) \geqslant \alpha$ within $\mathcal{G}$. We show that in fact $\chi(b) = \alpha$. Assume $\chi(b) > \alpha$. We have $b = pa$ for some $a \in \mathcal{G}[\alpha]$ and $p \in \widehat{P}$. But B is $\widehat{P}$-independent. This contradicts the fact that $p|1 \cdot b$ and 1 is evidently not divisible by p. Therefore, we have

$$[b] = Q(\alpha)b.$$

It remains to prove that $\mathcal{G} \subseteq \langle B \rangle$. Pick any nonzero $g \in \mathcal{G}$. There exist integers m and n such that $(m, n) = 1$ and $\chi(\frac{m}{n} g) = \alpha$. To see this we use the fact that $\chi(g) \in \mathbf{f}$. It is enough to make only finitely many changes to $\chi(g)$ to make it equivalent to α.

Equivalently, $\frac{m}{n} g \in \mathcal{G}[\alpha]$. We have $\frac{m}{n} g = \sum_{b \in B,\, r_b \in Q^{(P)}} r_b b$, by Lemma 4.4. By our assumption, $\chi(b) = \chi(\frac{m}{n} g) = \alpha$, for every $b \in B$. Obviously, $m | \frac{m}{n} g$ in $\mathcal{G}$. Therefore, by the definition of α and B, we have $m|b$ *in* $Q(\alpha)b$. Thus, there exist $x_b \in [b] = Q(\alpha)b$ such that $m x_b = b$. We can set $g = \sum_{b \in B} n r_b x_b$, where $n r_b x_b \in [b]$. This shows $\mathcal{G} = (B)_{Q(\alpha)}$. □

5. Effective content of S-independence, and Δ^0_3-categoricity

Theorem 5.1. *Every computably presentable homogeneous completely decomposable torsion-free abelian group is Δ^0_3-categorical.*

The proof of Theorem 5.1 is based on the lemma below. The proof of this lemma uses Theorem 4.10. The proof of Theorem 5.1 was sketched in [35].

Lemma 5.2. *Let $\mathcal{G} = \bigoplus_{i \in \omega} H$, where $H \leqq Q$, the type $\mathbf{t}(H)$ is $\mathbf{f}$, and $\alpha = (\alpha_0, \alpha_1, \ldots)$ is a characteristic of type $\mathbf{f}$. Let G_1 and G_2 be computable presentations of $\mathcal{G}$. Suppose that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases. Then there exists a Δ^0_n-isomorphism from G_1 onto G_2.*

We first prove Theorem 5.1, and then prove Lemma 5.2. We need to show that a given homogeneous completely decomposable group satisfies the hypothesis of Lemma 5.2 with $n = 3$.

Proof of Theorem 5.1. Let G be a computable presentation of $\mathcal{G} \cong \bigoplus_{i \in \omega} H$, where $H \leqslant Q$. Let α be a characteristic of type $\mathbf{t}(H)$ and $P = \{p_k\colon\ \alpha_k = \infty \text{ in } \alpha\}$. By Theorem 4.10 and Lemma 5.2, it suffices to construct a excellent $\widehat{P}$-basis of $G[\alpha]$ which is Σ^0_3.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At step n of the procedure, we do the following:

(1) Pick the n-th element g_n of $G[\alpha]$.
(2) Find an extension C_n of C_{n-1} in $G[\alpha]$ such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) $C_n \cup \{g_n\}$ is linearly dependent.

Let $G = \bigoplus_{i \in I} Re_i$, where $\chi(e_i) = \alpha$ and $R \cong H$. Observe that at stage n of the procedure we have $g_n \cup C_{n-1} \subset (\{e_0, \ldots, e_k\})_{Q^{(P)}}$, for some k. By Proposition 4.5, the needed extension denoted by C_n can be found.

It suffices to check that the construction is effective relative to $0''$. We use computable infinitary formulas in the proofs of the claims below. See [3] for a background on computable infinitary formulas.

By Theorem 4.10, we have $G[\alpha] \cong G_P$, where $P = \{p\colon\ p^\infty | h\}$ is a Π^0_2 set of primes.

Claim 5.3. *The group $G[\alpha]$ is c.e. in $0''$.*

Proof. Pick any $h \in G$ with $\chi(h) = \alpha$. By its definition, for every $g \in G$, the property $\chi(g) \geqslant \alpha$ is equivalent to

$$\bigwedge_{p\text{-prime}} \bigwedge_{k \in \omega} \bigl((\exists x) p^k x = h \to (\exists y) p^k y = g\bigr).$$

Therefore, the group $G[\alpha]$ is a Π^0_2-subgroup of G. $\square$

Claim 5.4. *There is a $0''$-computable procedure which decides if a given finite set $B \subseteq G[\alpha]$ is $\widehat{P}$-independent, uniformly in the index of B.*

Proof. It suffices to show that the property "B is a $\widehat{P}$-independent set in $G[\alpha]$" can be expressed by a Π^0_2 infinitary computable formula in the signature of abelian groups with parameters elements from B.

Note that in general $P \in \Pi^0_2$. By Claim 5.3, the group $G[\alpha]$ is a Π^0_2-subgroup of G. Thus, the condition "B is a $\widehat{P}$-independent set in $G[\alpha]$" seems to be merely Π^0_3:

$$\bigwedge_{\bar{m} \in Z^{<\infty}} \bigwedge_{p\text{-prime}} \left(\left[p \notin P \wedge (\exists x)\left(x \in G[\alpha] \wedge px = \sum_{b \in B} m_b b\right)\right] \to \bigwedge_b p | m_b\right).$$

The idea is to substitute the Σ^0_3 formula $(\exists x)(x \in G[\alpha] \wedge px = \sum_{b \in B} m_b b)$ by an equivalent Σ^0_2 one, using a non-uniform parameter $c \in G$ such that $\chi(c) = \alpha$. More specifically, we are going to show that for every $p_v \notin P$, the formula

$$(\exists x)\left(x \in G[\alpha] \wedge p_v x = \sum_{b \in B} m_b b\right)$$

is equivalent to

$$(\exists k)(\exists y \in G)\left(\alpha_v < k \wedge p_v^k y = \sum_{b \in B} m_b b\right),$$

where α_v is the v-th component of α corresponding to p_v and

$$\alpha_v < k \quad \Leftrightarrow \quad \neg(\alpha_v \geqslant k) \quad \Leftrightarrow \quad \neg(\exists \xi)\bigl(p_v^k \xi = c\bigr).$$

Suppose there is $x \in G[\alpha]$ such that $p_v x = \sum_{b \in B} m_b b$. Since $h_v(x) \geqslant \alpha_v$, we have $p_v^{\alpha_v} y = x$ and $p_v^{\alpha_v+1} y = p_v x$, for some $y \in G$, so we can set $k = \alpha_v + 1$. For the converse, suppose there exist such k and y. Then $p_v x = p_v^k y$ for $x = p_v^{k-1} y$. We have $k > \alpha_v$, and therefore $(k-1) \geqslant \alpha_v$. But $h_v(x) \geqslant (k-1)$ because $x = p_v^{k-1} y$ is divisible by $k-1$, and thus $h_v(x) \geqslant \alpha_v$. The characteristic of x differs from the characteristic of y only at the position for the prime p_v. Thus, for every $w \neq v$,

$$h_w(x) = h_w\left(p_v^k y\right) = h_w\left(\sum_{b \in B} m_b b\right) \geqslant \alpha_w,$$

since $\sum_{b \in B} m_b b \in G[\alpha]$. Therefore, $\chi(x) \geqslant \alpha$ and $x \in G[\alpha]$. $\square$

By Claim 5.3 and Claim 5.4, the procedure is computable relative to $0''$. $\square$

Proof of Lemma 5.2. Recall that G_1 and G_2 are computable presentations of $\mathcal{G}$ such that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ_n^0 excellent $\widehat{P}$-bases. We need to show that there exists a Δ_n^0-isomorphism from G_1 onto G_2. Let B_1 and B_2 be excellent $\widehat{P}$-bases of G_1 and G_2, respectively.

Observe that the group $Q(\alpha)$ is isomorphic to a c.e. additive subgroup R of $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$. To see this pick h with $\chi(h) = \alpha$ non-uniformly, and then apply Theorem 2.7 to the group $[h]$. By Theorem 4.10, we have

$$G_1 = \bigoplus_{b \in B_1} Rb \cong G_2 = \bigoplus_{b' \in B_2} Rb'.$$

To build a Δ_n^0-isomorphism from G_1 to G_2 first define the map from B_1 onto B_2 using a standard back-and-forth argument. Then extend it to the whole G_1 using the fact that $r \cdot b$ can be found effectively and uniformly, for every $r \in R$ and $b \in B_1$. $\square$

By Proposition 3.6 and Remark 3.7, "computable presentation of G_P" can be equivalently understood as "computable presentation of the group G_P" or "computable presentation of the $Q^{(P)}$-module G_P". Before we turn to a more detailed study of Δ_2^0-categorical completely decomposable groups, we prove a fact about excellent $\widehat{P}$-bases of the group G_P which is of an independent interest for us:

Theorem 5.5. *If a computable presentation of G_P has a Σ_2^0-basis which generates it as a free $Q^{(P)}$-module, then this presentation possesses a Π_1^0-basis which generates it as a free $Q^{(P)}$-module.*

Proof. Recall that, by Lemma 4.4, a basis generates G_P as a free $Q^{(P)}$-module if and only if this basis is an excellent $\widehat{P}$-basis. The proof of the theorem is based on Lemma 4.4 and the short technical lemma below.

Lemma 5.6. *Suppose $\{e_i\colon\ i \in \omega\} \subset G_P$ is such that $G_P = \bigoplus_{i \in \omega} Q^{(P)} e_i$, and suppose $\{b_1, \dots, b_k\} \subset G_P \setminus \{0\}$. For any integer $m, k \neq 0$, the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent if and only if $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + me_0\}$ is $\widehat{P}$-independent. Furthermore, $(B)_{Q^{(P)}} = (B_m)_{Q^{(P)}}$, for every m.*

Note that for the second part of Lemma 5.6 *we do not assume that B is $\widehat{P}$-independent.*

Proof of Lemma 5.6. Suppose $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. We show that $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + me_0\}$ is $\widehat{P}$-independent as well.

Pick an arbitrary $p \in \widehat{P}$. Suppose that p divides $g = n_0 e_0 + \sum_{1 \leqslant i \leqslant k-1} n_i b_i + n_k(b_k + me_0) = (n_0 + n_k m)e_0 + \sum_{1 \leqslant i \leqslant k} n_i b_i$. Recall that the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. Therefore, $p | n_i$, for every $1 \leqslant i \leqslant k$. As a consequence, p divides $n_0 e_0 = g - n_k m e_0 - \sum_{1 \leqslant i \leqslant k} n_i b_i$. By our assumption on the element e_0, we have $p | n_0$. $\square$

Suppose that $E = \{e_0, e_1, \ldots\}$ is a Σ^0_2 excellent $\widehat{P}$-basis of $G = \bigoplus_{i \in \omega} Q^{(P)} e_i = \{g_0 = 0, g_1, \ldots\}$ which is a computable group. We fix a computable relation R such that $x \in E$ if and only if $(\exists^{<\infty} y) R(x, y)$. We build a co-c.e. set of elements B such that the following requirements are met:

R_0: $e_0 \in B$;
R_j: if $g_j = e_k$ for some k then B contains exactly one element of the form $(e_k + m e_0)$.

We also require that the only elements that enter B are due to one of these requirements. There is no priority order on the requirements.

We first show that if all the requirements are met, then the set B is an excellent $\widehat{P}$-basis of G. Assume R_j is met, for every j. It follows that for every k there exists m such that $e_k + m e_0 \in B$. Also, if B contains two elements of the form $e_k + m e_0$ and $e_k + n e_0$, then necessarily $n = m$. It remains to show that B is an excellent $\widehat{P}$-basis of G. Note that, if B is not $\widehat{P}$-independent, then there is a finite subset B_0 of B which is not $\widehat{P}$-independent. By (a multiple application of) Lemma 5.6, this contradicts the choice of $E = \{e_0, e_1, \ldots\}$. It remains to apply the second part of Lemma 5.6 and see that the $Q^{(P)}$-spans of B and E coincide.

All strategies in the construction will share the same global restraint. More specifically, in the construction the strategies will put restraints onto certain elements of the group. The desired set B will consist of elements which eventually become forever restrained by the strategies.

Strategy for R_0. Permanently restrain e_0.
Strategy for R_j, $j > 0$. If R_j currently has no witness then pick a witness c_j which is equal to $g_j + m e_0$, where m is the least such that $g_j + m e_0$ is not restrained and is not yet enumerated into $\overline{B}$. Declare c_j restrained (thus, c_j is now our witness, and our current guess is $c_j \in B$). If c_j is the n-th element of the group, $c_j = g_n$, then enumerate each g_x with $x < n$ into $\overline{B}$ unless g_x is already in $\overline{B}$ or is restrained. If, at a later stage, a fresh y is found such that $R(g_j, y)$ holds, then enumerate $g_j + m e_0$ into $\overline{B}$, and initialize R_j by making c_j undefined.

Construction.
Stage s. Let R_j, $j \leqslant s$, act according to their instructions.
End of construction.

The set B consists of elements which eventually become forever restrained by strategies. Also note that each element of the group can be restrained at most once. Thus, the set $\overline{B}$ is c.e.

To see why R_j is met note that the requirement eventually puts a permanent restraint on its witness $g_j + m e_0$ if an only if $(\exists^{<\infty} y) R(g_j, y)$. This is the same as saying that $g_j = e_k$, for some k. $\square$

6. Semi-low sets, and Δ^0_2-categoricity

Recall that a set A is semi-low if the set $H_A = \{e\colon\ W_e \cap A \neq \emptyset\} = \{e\colon\ W_e \nsubseteq \overline{A}\}$ is computable in $\emptyset'$.

Theorem 6.1. *A computably presentable completely decomposable abelian group G is Δ^0_2-categorical if and only if G is isomorphic to G_P where $\widehat{P}$ is semi-low.*

The proof of this theorem is split into several parts. Each part corresponds to a different hypothesis on the isomorphism type of G. Different cases will need different techniques and strategies.

Proof of Theorem 6.1. We need the following technical notion:

Definition 6.2. Let $\alpha = (h_i)_{i \in \omega}$ be a c.e. characteristic (see Definition 3.1), and let $h_{i,s}$ be its non-decreasing uniform computable approximation: $h_i = \sup_s h_{i,s}$, for every i. We say that α has *a computable settling time* if there is a (total) computable function $\psi : \omega \to \omega$ such that

$$h_i = \begin{cases} h_{i,\psi(i)}, & \text{if } h_i \text{ is finite,} \\ \infty, & \text{otherwise,} \end{cases}$$

for every i. We also say that ψ is a computable settling time for $(h_{i,s})_{i,s\in\omega}$.

This is the same as saying that, given i, there exists an effective (and uniform) way to compute a stage s after which the approximation of h_i either does not increase, or increases and tends to infinity. Note that this is the property of a characteristic, not the property of some specific computable approximation. Indeed, given an approximation of α having a computable settling time, we can define a computable settling time for any other computable approximation of α. Furthermore, as can be easily seen, this is a type-invariant property. Thus, we can also speak of types having computable settling times.

If a homogeneous completely decomposable group G of type $\mathbf{f}$ is computable, then $\mathbf{f}$ is c.e. (see Proposition 3.3). Suppose that G is a computable homogeneous completely decomposable group of type $\mathbf{f}$, and let $\alpha = (h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We consider the cases:

(1) The type $\mathbf{f}$ of G has no computable settling time. In this case G is not Δ^0_2-categorical by Proposition 6.5. Observe that if $\mathbf{f}$ has no computable settling time then the set $\mathit{Fin}(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ has to be infinite (see, e.g., Proposition 3.6). Thus, G cannot be isomorphic to G_P, for a set of primes P.
(2) The type $\mathbf{f}$ of G has a computable settling time, $\mathit{Fin}(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is empty (finite), and the set $\{i\colon\ h_i = 0\}$ is semi-low. In other words, the group G is isomorphic to G_P with $\widehat{P}$ semi-low. In this case G is Δ^0_2-categorical, by Proposition 6.3 below.
(3) The type $\mathbf{f}$ of G has a computable settling time, the set $\mathit{Fin}(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is empty (finite), and the set $\{i\colon\ h_i = 0\}$ is not semi-low. Here G is again isomorphic to G_P, but in this case G is not Δ^0_2-categorical, by Proposition 6.6 below.
(4) The type $\mathbf{f}$ of G has a computable settling time, and the set $\mathit{Fin}(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is infinite and not semi-low. As in the above case,[1] G is not Δ^0_2-categorical, by Proposition 6.6.
(5) The type $\mathbf{f}$ of G has a computable settling time, and the set $\mathit{Fin}(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is infinite and semi-low. The group is not Δ^0_2-categorical, by Proposition 6.7 below.

We first discuss why case (3) and case (4) above can be collapsed into one case. First, define $\mathit{Inf}(\alpha) = \{i\colon\ h_i = \infty\}$ and $V = \{i\colon\ 0 < h_{i,\psi(i)} < \infty\}$, where ψ is a computable settling time for α. Note that V is c.e. Evidently, $\overline{\mathit{Inf}(\alpha)} = \mathit{Fin}(\alpha) \cup \{i\colon\ h_i = 0\}$ and $\mathit{Fin}(\alpha) = \overline{\mathit{Inf}(\alpha)} \cap V$. We claim that "$\mathit{Fin}(\alpha)$ is not semi-low" implies "$\overline{\mathit{Inf}(\alpha)}$ is not semi-low". We assume that $\overline{\mathit{Inf}(\alpha)}$ is semi-low and observe that $\{e\colon\ W_e \cap \mathit{Fin}(\alpha) \neq \emptyset\} = \{e\colon\ W_e \cap V \cap \overline{\mathit{Inf}(\alpha)} \neq \emptyset\} = \{e\colon\ W_{s(e)} \cap \overline{\mathit{Inf}(\alpha)} \neq \emptyset\}$ for a computable function s. Therefore, $H_{\mathit{Fin}(\alpha)} \leqslant_m H_{\overline{\mathit{Inf}(\alpha)}} \leqslant_T \emptyset'$, as required.

Therefore, cases (3) and (4) are both collapsed into

(3′) If $\mathbf{f}$ has a computable settling time and $\overline{\mathit{Inf}(\alpha)}$ is not semi-low, then G is not Δ^0_2-categorical.

Now we state and prove the propositions which cover all the cases above.

Recall that, by Proposition 3.6, the group G_P has a computable presentation as a group (module) if and only if P is c.e.

Proposition 6.3. *If $\widehat{P}$ is semi-low (and co-c.e.) then G_P is Δ^0_2-categorical.*

Proof. The proof may be viewed as a simpler version of the proof of Theorem 5.1. Let $G = \{g_0 = 0, g_1, \ldots\}$ be a computable copy of G_P. By Lemma 3.8, it is enough to build a Σ^0_2 excellent $\widehat{P}$-basis of G.

[1] We distinguish these two cases only because these cases correspond to (algebraically) different types of groups. We discuss a bit later why these cases are essentially not different.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At stage n of the construction, we do the following:

(1) Pick the n-th element g_n of G.
(2) Find an extension C_n of C_{n-1} in G such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) $C_n \cup \{g_n\}$ is linearly dependent.

The algebraic part of the verification is the same as in Theorem 5.1 (and is actually simpler). Thus, it is enough to show that (a) in (2) above can be checked effectively and uniformly in $\emptyset'$. Given a finite set F of elements of G, define a c.e. set V consisting of primes which could potentially witness that F is $\widehat{P}$-dependent:

$$V = \left\{p\colon \bigvee_{\overline{m} \in Z^{card(F)}} \left[p \,\Big|\, \Big(\sum_{g \in F} m_g g\Big) \wedge \Big(\bigvee_{g \in F} p \nmid m_g\Big)\right]\right\}.$$

The c.e. index of V can be obtained uniformly from the index of F. It can be easily seen from the definition of $\widehat{P}$-independence that

$$V \cap \widehat{P} = \emptyset \quad \text{if and only if} \quad F \text{ is } \widehat{P}\text{-independent}.$$

By our assumption on $\widehat{P}$, this can be decided effectively in $\emptyset'$. $\square$

Fix a computable listing $\{\Phi_e(x, y)\}_{e \in \omega}$ of all partial computable functions of two arguments. We say that $\lim_s \Phi_e(x, s)$ exists if $\Phi_e(x, s)\downarrow$ for every e and s and the sequence $(\Phi_e(x, s))_{s \in \omega}$ stabilizes. In the upcoming propositions we will use the following:

Notation 6.4. Fix an effective listing $\{\Psi_e(x, s)\}_{e \in \omega}$ of total computable functions of two arguments satisfying the property:

$$\Big(\lim_s \Phi_e(x, s) \text{ exists}\Big) \quad \Rightarrow \quad \Big(\lim_s \Phi_e(x, s) = \lim_s \Psi_e(x, s)\Big),$$

for every x and e. (We may assume that $\Psi_e(x, 0) = 0$, for every x and e.)

Proposition 6.5. *Suppose that the type* $\mathbf{f}$ *of a computably presentable* $G = \bigoplus_{i \in \omega} H$ *has no computable settling time. Then* G *is not* Δ^0_2*-categorical.*

Proof idea. Let $\alpha = (h_i)_{i \in \omega}$ be a characteristic of type $\mathbf{f}$. We build two computable groups, A and B, both isomorphic to G. The group A is a “nice” copy of G. The group B is a “bad” copy of G in which the e-th elementary direct component is used to defeat the e-th potential Δ^0_2-isomorphism from B onto A.

The first main idea of the strategy uses Baer's theory of types. We wait for the e-th potential isomorphism to converge on some specifically chosen element b_e from the e-th elementary direct component of B. We pick a fresh number j so large that, if the e-th potential isomorphism is indeed an isomorphism, the characteristic $\chi(b_e) = (d_i)_{i \in \omega}$ of b_e and the characteristic $\alpha = (h_i)_{i \in \omega}$ have to be equal starting from the j-th position. We may choose such a number j using that A is “nice” (to be explained in more detail). From this moment on, make sure $d_{k,s} = h_{k,s} - 1$ for $k \geqslant j$ least such that $h_{k,t} > 0$, where t is the current stage of the construction and $s \geqslant t$. By the choice of $\mathbf{f}$, such a position k can be found. Note that the e-th potential isomorphism is merely a (partial) Δ^0_2 function, and at a later stage it may output a new potential image of b_e. In this case we make $d_{k,s} = h_{k,s}$ and repeat the strategy.

The strategy would work if we had no symbols ∞ in $\mathbf{f}$. If we have ∞ on $\mathbf{f}$, then it may happen that

$$h_k = \lim_t h_{k,t} = \infty,$$

for the k we pick at the final iteration of the strategy (if the strategy iterates infinitely often then we win). In this case the strategy fails because both $h_{k,s}$ and $d_{k,s} = h_{k,s} - 1$ tend to infinity.

The second main idea is to pick a new fresh position k_1 for which $h_{k_1,s} > 0$ if we see $h_{k,s} > h_{k,t}$ at a later stage s. We may keep iterating this strategy defining k_2 when both h_k and h_{k_1} increase, etc. Nonetheless, this strategy is not sufficient if

$$h_{k_i} = \lim_t h_{k_i,t} = \infty$$

for every i.

The third main idea uses the notion of computable settling time. More specifically, each time we pick a new position k_i as described above, we additionally attempt to define a computable settling time ψ for α. If we have to make one more iteration as described in the previous paragraph, we set $\psi(k_i) = t$. We also define ψ on arguments between k_i and k_{i+1} (to be explained formally in the construction).

We keep introducing k_{i+1}, k_{i+2} etc. This process never terminates only if every position we pick corresponds to ∞ in α. Thus, we will succeed in defining a computable settling time for $\mathbf{f}$, contradicting the choice of $\mathbf{f}$ (to be explained in more detail). Therefore, we eventually pick a position k_j such that $h_{k_j} < \infty$. The groups A and B are both isomorphic to G by Theorem 2.7, because the characteristic of b_e belongs to $\mathbf{f}$. (Several minor technical details have not been mentioned in this sketch.)

Proof of Proposition 6.5. In the construction below we identify elements of A and B and the corresponding elements of ω. It suffices to build two computable presentations, A and B, of the group $G = \bigoplus_{i\in\omega} H$, and meet the requirements:

R_e: $\lim_t \Psi_e(b_e, t)$ exists $\Rightarrow$ $\lim_t \Psi_e(x, t)$ is not an isomorphism from B to A.

The nonzero element b_e is a witness for the R_e strategy below. More specifically, we enumerate $A = \bigoplus_{n\in\omega} Ha_n$ and $B = \bigoplus_{e\in\omega} C_e b_e$ in such a way that the sets $\{a_n\colon n \in \omega\}$ and $\{b_e\colon e \in \omega\}$ are computable. Let $(h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. Fix a computable approximation $(h_{i,s})_{i,s\in\omega}$ of $(h_i)_{i\in\omega}$ such that (1) $h_{i,s} \leqslant h_{i,s+1}$, and (2) $h_i = \lim_s h_{i,s}$, for every i and s.

We make sure $\chi(a_n) = (h_i)_{i\in\omega}$, for every n, while the characteristic $\chi(b_e) = (d(e)_i)_{i\in\omega}$ of b_e will be merely equivalent to $(h_i)_{i\in\omega}$, for each e (thus, $C_e \cong H$, for each e).

The construction is injury-free, and we do not need any priority order on the strategies.

For every e, the strategy for R_e defines its own computable function ψ_e which[2] is an attempt to define a computable settling time for $(h_i)_{i\in\omega}$. To define ψ_e the strategy uses the sequence $(k_{e,i})_{i\in\omega}$ (to be defined in the construction).

Strategy for R_e. If at a stage s of the construction the parameter $k_{e,0}$ is undefined then:

(1) Compute $\Psi_e(b_e, s)$. From this moment on, the strategy is always waiting for $t > s$ such that $\Psi_e(b_e, t) \neq \Psi_e(b_e, s)$. As soon as such a t is found, R_e initializes by making all its parameters undefined and also making $d(e)_{j,t} = h_{j,t}$ for every j we have ever seen so far.

[2] Since it will be clear from the construction at which stage ψ_e is defined (if ever), we omit the extra index t in $\psi_{e,t}$ and write simply ψ_e. We omit the index t for parameters $k_{e,i,t}$ as well.

(2) Let $a \in A$ be such that $a = \Psi_e(b_e, s)$. Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let j be a fresh large index such that (1) the prime p_j does not occur in the decompositions of the coefficients c and c_n, (2) $h_{j,s} > 0$, and (3) $d(e)_{j,s} < h_{j,s}$.
(3) Once j is found,[3] declare $\psi_e(j) = s$. From this moment on, make sure $d(e)_{j,t} = h_{j,t} - 1$ for every $t \geqslant s$, unless the strategy initializes. Set $k_{e,0} = j$, and proceed.

Now assume that the parameters $k_{e,0}, \dots, k_{e,y}$ have already been defined by the strategy. We also assume that $\psi_e(i)$ has already been defined for each i such that $k_{e,0} \leqslant i \leqslant \max\{k_{e,x}\colon\ 0 \leqslant x \leqslant y\}$. Assume also that $k_{e,y}$ was first defined at stage $u < s$. Then do the following:

I. Wait for a stage $t \geqslant s$ (of the construction) such that either (a) $h_{i,t} > h_{i,s}$ for some i such that $k_{e,0} \leqslant i \leqslant \max\{k_{e,x}\colon\ 0 \leqslant x \leqslant y\}$ and $i \notin \{k_{e,0}, \dots, k_{e,y}\}$, or (b) $h_{i,u} < h_{i,t}$ for each $i \in \{k_{e,0}, \dots, k_{e,y}\}$. While waiting, make $d(e)_{j,r} = h_{j,r}$ (r is the current stage of the construction), where $j \leqslant r$ and $j \notin \{k_{e,0}, \dots, k_{e,y}\}$.
II. If (a) holds for some i, then set $k_{e,(y+1)} = i$. If (b) holds, then let i be a fresh large index such that (1) $h_{i,t} > 0$, and (2) $d(e)_{i,t} < h_{i,t}$, and set $k_{e,(y+1)} = i$. In this case also define $\psi_e(j)$ to be equal to the current stage for every j such that $\max\{k_{e,x}\colon\ 0 \leqslant x \leqslant y\} < j \leqslant k_{e,(y+1)}$. Then proceed to III.
III. Set $d(e)_{i,t} = h_{i,t} - 1$ at every later stage t, where $i = k_{e,(y+1)}$, unless the strategy initializes.

End of strategy.

Construction. At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively. Initialize R_e, for all e.

At stage s, let strategies R_e, $e \leqslant s$, act according to their instructions. If R_e acted at the previous stage, then return to its instructions at the position it was left at the previous stage.

Make $\chi(a_n) = (h_{i,s})_{i\in\omega}$ in A_s for every $n \leqslant s$, and $\chi(b_e) = (d(e)_{i,s})_{i\in\omega}$ in B_s for every $e \leqslant s$, by making a_n and b_e divisible by corresponding powers of primes.
End of construction.

Verification. For each e, the following cases are possible:

(1) $\lim_s \Psi_e(b_e, s)$ does not exist. In this case the strategy initializes infinitely often. By the way the strategy is initialized, the characteristic of b_e is identical to α.
(2) $\lim_s \Psi_{e,s}(b_e, s)$ exists and is equal to $\Psi_e(b_e, l)$. The domain of ψ_e should be co-infinite. For if it was co-finite, then α would have a computable settling time. Therefore, there is a parameter $k_{e,y}$ such that the $k_{e,y}$-th position in α is finite. Thus, the strategy ensures $\lim_s \Phi_{e,s}(b_e, s)$ is not an isomorphism since the characteristic of b_e and α differ at $k_{e,y}$-th position. Therefore, α differs from $\chi(b_e)$ in at most finitely many positions, and the differences are finitary.

In both cases $\chi(b_e)$ is equivalent to α. By Theorem 2.7, $A \cong B \cong G$. $\square$

Recall that cases (3) and (4) were both reduced to:

Proposition 6.6. *Let G be computable homogeneous completely decomposable abelian group of type* $\mathbf{f}$*, and suppose $\alpha = (\sup_s h_{i,s})_{i\in\omega}$ in* $\mathbf{f}$ *has computable settling time ψ. Furthermore, suppose $\overline{Inf(\alpha)}$ is not semi-low. Then G is not Δ^0_2-categorical.*

Proof idea. We build two computable groups, A and B, both isomorphic to G. The group A is a "nice" copy of G. The group $B = \bigoplus_{e\in\omega} \bigoplus_{n\in\omega} C_{e,n} b_{e,n}$ is a "bad" copy of G in which the e-th direct component is used to defeat the e-th potential Δ^0_2-isomorphism from B onto A.

Recall that $Inf(\alpha)$ is a c.e. set. Given e, we attempt to define a functional $\Gamma(e, n, s)$ such that $H_{\overline{Inf(\alpha)}}(n) = \lim_s \Gamma(e, n, s)$. For every n, we pick an element $b_{e,n}$ in B and attempt to destroy the e-th

[3] We may assume that at stage s such an index j can be found, otherwise we speed up the approximation $(h_{i,s})_{i,s\in\omega}$ during the construction.

potential Δ_2^0-isomorphism from B to A. We start by setting $\Gamma(e,n,0)=0$. We wait for j to appear in $W_{n,s}\setminus Inf(\alpha)_s$. If we never see such a j, then our attempt to define $\Gamma(e,n,s)$ is successful. If we find such a j, make $b_{e,n}$ divisible by a large power of p_j destroying the potential isomorphism (this power depends on our current guess on the isomorphic image of $b_{e,n}$ in A). We will set $\Gamma(e,n,t)=1$ only if the e-th potential isomorphism changes on $b_{e,n}$ at a later stage t. We make $\Gamma(e,n,r)=0$ as soon as j enters $Inf(\alpha)$, and then we start waiting for a new fresh number to show up in $W_n\setminus Inf(\alpha)$. If we see such a number then we repeat the above strategy with this number in place of j.

Our attempt to define $\Gamma(e,n,s)$ necessarily fails for at least one index n. Therefore, the e-th potential isomorphism will be defeated at the element $b_{e,n}$. Algebra is sorted out using Theorem 2.7.

Note that the algebraic strategy above differs from the one we used in Proposition 6.5. More specifically, we make elements divisible instead of keeping elements non-divisible. This strategy could not be used in Proposition 6.5, because it would not be consistent with the infinitary outcome (the case when the e-th potential isomorphism changes infinitely often). We will see that this is not a problem here.

Proof of Proposition 6.6. We build two computable copies of G by stages. Recall that the first copy $A=\bigoplus_i Ha_i$ is a "nice" copy with $\chi(a_i)=\alpha$, for every i. The second ("bad") copy $B=\bigoplus_{e\in\omega}\bigoplus_{n\in\omega}C_{e,n}b_{e,n}$ is built in such a way that $\chi(b_{e,n})$ is equivalent to α, for every e and n.

Recall Notation 6.4. It suffices to meet the requirements:

R_e: $(\forall n)\lim_t \Psi_e(b_{e,n},t)$ exists $\Rightarrow$ $\lim_t \Psi_e(x,t)$ is not an isomorphism from B to A.

The strategy for R_e initially attempts to define a total $0'$-computable function Γ such that $\Gamma(n)=0$ iff $W_n\subseteq Inf(\alpha)$. If we succeeded, this would imply

$$H_{\overline{Inf(\alpha)}}=\{n\colon W_n\cap\overline{Inf(\alpha)}\neq\emptyset\}=\{n\colon W_n\nsubseteq Inf(\alpha)\}\leqslant_T\emptyset',$$

contradicting the hypothesis. In the following, we write I in place of $Inf(\alpha)$. Also, we omit e in $\Gamma(e,n,s)$ and write simply $\Gamma(n,s)$. We also assume at most one number can be enumerated into W_n at every stage. We split R_e into substrategies $R_{e,n}$, $n\in\omega$:

Substrategy $R_{e,n}$. Permanently assign the element $b_{e,n}$ to $R_{e,n}$. Suppose that the strategy becomes active first time at stage s of the construction. Then:

(1) Start by setting $\Gamma(n,s)=0$ (we may suppose that $\Gamma(n,j)=0$, for every $j<s$). At a later stage t, we define $\Gamma(n,t)$ to be equal to $\Gamma(n,t-1)$, unless we have a specific instruction not to do so.
(2) Wait for a stage $t>s$ and a number $j\in W_{n,t}\setminus I_t$. (Recall that we assume that at most one number can be enumerated into W_n at a stage.)
(3) We let $p=p_j$ with $j\in W_{n,t}\setminus I_t$ at a later stage t. Find $a\in A_t$ such that $a=\Psi_e(b_e,t)$ (recall that the enumeration of A is controlled by us). Find integers c_n and c such that $ca=\sum_n c_n a_n$. Let k be a fresh large natural number such that (i) the prime $p=p_j$ has power at most $[k/2]$ in the decompositions of the coefficients c and c_n, and (ii) $h_{j,\psi(j)}<[k/2]$, where ψ is the computable settling time. Note that (i) and (ii) imply k is so large that p^k does not divide $a=\Psi_e(b_{e,n},t)$ within A, unless $j\in I_t$. Make $b_{e,n}$ divisible by p^k within B.
Wait for one of the two things to happen:
    I. (I changes first). We see $j\in I_u$ at a later stage $u>t$, and $\Psi_e(b_{e,n},v)=\Psi_e(b_{e,n},t)$ for each $v\in(t,u]$. We return to (2) with u in place of s.
    II. (Ψ_e changes first). We see $\Psi_e(b_{e,n},u)\neq\Psi_e(b_{e,n},t)$ for $u>t$, and $j\in W_{n,v}\setminus I_v$ for each $v\in(t,u]$. Then set $\Gamma(n,u)=1$ and start waiting for a stage $w>u$ such that $j\in I_w$. If such a stage w is found, then we set $\Gamma(n,w)=0$ and go to (2) with w in place of s (and we do nothing, otherwise).

End of strategy.

Construction. At stage 0, start enumerating A and B as free abelian groups over $\{a_i\}_{i\in\omega}$ and $\{b_{e,n}\}_{e,n\in\omega}$.

At stage s, let strategies $R_{e,n}$, $e, n \leqslant s$, act according to their instructions. If $R_{e,n}$ acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make $\chi(a_i) = \alpha = (h_j)_{j\in\omega}$ in A for every i. For every $e, n \in \omega$, make $\chi_j(b_{e,n}) = h_j$ in B for every j except at most one position, according to the instructions of $R_{e,n}$. We do so by making a_i and $b_{e,n}$ divisible by corresponding powers of primes.

End of construction.

Verification. By Theorem 2.7, $A \cong B \cong G$. Assume that $\lim_s \Psi_{e,s}(b_{e,n}, s)$ exists for every n (thus, II does not get visited infinitely often). Given n, consider the cases:

- $R_{e,n}$ eventually waits forever at substage (2). Then $\lim_s \Gamma(n, s) = 0$ and $W_n \subseteq I$. Thus, we have a correct guess about $H_{\overline{Inf(\alpha)}}$.
- $R_{e,n}$ visits I of (3) again and again from some point on. Then $\lim_s \Gamma(n, s) = 0$ and $W_n \subseteq I$, and we again have a correct guess about $H_{\overline{Inf(\alpha)}}$.
- $R_{e,n}$ eventually waits forever at substage (3). Then $b_{e,n}$ witnesses that $\lim_s \Psi_e(b_{e,n}, s)$ is not an isomorphism.

There should be at least one n for which $\lim_s \Gamma(n, s) \neq H_{\overline{Inf(\alpha)}}(n)$. Therefore, for at least one n, the strategy $R_{e,n}$ eventually waits forever at substage (3). Thus, R_e is met. $\square$

Proposition 6.7. *If the type* $\mathbf{f}$ *of a computable homogeneous completely decomposable group G has a computable settling time, and $Fin(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is infinite and semi-low for $\alpha = (\alpha_i)_{i\in\omega}$ of type* $\mathbf{f}$*, then G is not Δ^0_2-categorical.*

Proof idea. We combine the algebraic strategy from Proposition 6.5 and the guessing procedure based on the hypothesis $Fin(\alpha) = \{i\colon\ 0 < h_i < \infty\}$ is semi-low. As before, we are building two computable copies, A and B, of G.

If $Fin(\alpha)$ were an infinite computable set, then the algebraic strategy would be rather straightforward. To destroy the e-th potential Δ^0_2-isomorphism from B to A, we pick a large $j \in Fin(\alpha)$ and make the witness $b_e \in B$ *not divisible* by p_j. If the potential isomorphism changes at a later stage, we make $h_i(b_e) = \alpha_i$ and repeat the strategy for another fresh and large $i \in Fin(\alpha)$. We have already discussed a similar algebraic strategy in the idea of proof of Proposition 6.5 (the case with no ∞'s in α).

However, $Fin(\alpha)$ is merely semi-low. Recall that the type has a computable settling time. Therefore, we can produce a computable approximation $(h_{i,s})_{i,s\in\omega}$ of α such that, for every i, either $\alpha_i = h_{i,0}$ or $\alpha_i = \infty$. We focus on the computable set $N = \{i\colon\ h_{i,0} \neq 0\} = Inf(\alpha) \cup Fin(\alpha)$. Note that $Inf(\alpha) = \{i\colon\ \alpha_i = \infty\}$ is c.e.

Imagine the e-th potential Δ^0_2-isomorphism has settled on its witness $b_e \in B$ (if it never settles we win). To successfully run the algebraic strategy, we need to find at least one $i \in Fin(\alpha)$. We find a fresh large $i \in N$ and keep b_e not divisible by p_i. We can do so because i is so large that b_e has not been declared divisible by $p_i^{h_{i,0}}$ yet. At the same time we start enumerating a c.e. set first setting $W = \emptyset$, and ask if $W \cap Fin(\alpha) = \emptyset$ (recall that the guessing procedure is Δ^0_2). We do nothing and wait until we get the answer $W \cap Fin(\alpha) = \emptyset$. Note that we should eventually see this answer, otherwise we get a contradiction by keeping W empty. Then we enumerate i into W. We do not make b_e divisible by any further prime until we see:

(1) i enters $Inf(\alpha)$. Then we pick next least $j \in N$, enumerate j into W, and repeat the strategy keeping b_e untouched.

(2) The current guess becomes $W \cap Fin(\alpha) \neq \emptyset$. We allow the construction to continue building the elementary component corresponding to b_e but keep b_e not divisible by p_i. If i never enters $Inf(\alpha)$ we win. If at a later stage i enters $Inf(\alpha)$, we wait until our guess is $W \cap Fin(\alpha) = \emptyset$. Again, it should eventually happen, otherwise we get a contradiction by not changing W. Then we make b_e infinitely divisible by p_i, pick a large fresh $v \in N$, enumerate v into W, and repeat the whole strategy with v in place of i (again, keep b_e untouched etc.).

Note that we eventually reach (2) with some $j \in W$, and either j never enters $Inf(\alpha)$ or we change our guess on $W \cap Fin(\alpha)$. In the latter case will reach (2) again with another number, and either win or change the guess once more. We cannot change the guess infinitely often, because $Fin(\alpha)$ is semi-low. Thus, eventually the algebraic strategy succeeds.

In the formal construction each strategy defines its own *sequence* of c.e. sets. Every set from the sequence corresponds to a potential image of b_e, which can be changed at a later stage. If the image changes, we start enumerating the next set from the e-th sequence. Since the construction is effective and uniform, we may assume that the indexes of these c.e. sets are listed by a computable function, and the index of this function is given ahead of time. We give all details in the formal proof below.

Proof of Proposition 6.7. Let Γ be a computable function such that

$$Fin(\alpha) \cap W_n = \lim_s \Gamma(n, s).$$

As in the proof of Proposition 6.5, we are building two computable copies,

$$A = \bigoplus_{n \in \omega} Ha_n \quad \text{and} \quad B = \bigoplus_{e \in \omega} C_e b_e,$$

of G. We make $\chi(a_n) = \alpha$ and $\chi(b_e) = (d(e))_{i \in \omega} \simeq \alpha$, for every n and e. Recall Notation 6.4. The requirements are:

R_e: If $\lim_t \Psi_e(b_e, t)$ exists, then $\lim_t \Psi_e(x, t)$ is not an isomorphism from B to A.

For every e, the strategy for R_e will enumerate its own sequence of c.e. sets. The indexes for the sets are listed by a computable function g of two arguments:

$$\{W_{g(e,s)}\}_{s \in \omega}.$$

Let $(h_{i,s})_{i,s \in \omega}$ be a computable approximation of α such that, for every i, either $\alpha_i = h_{i,0}$ or $\alpha_i = \infty$. Also, let $n(0), n(1), \dots$ be an effective increasing enumeration of the infinite computable set $N = \{i\colon\ h_{i,0} \neq 0\}$.

The strategy for R_e. Suppose $s = 0$ or $\Psi_e(b_e, s) \neq \Psi_e(b_e, s-1)$. Do the following substeps:

(1) Make $\chi(b_e) = (d(e))_{i \in \omega}$ and α equal at all positions seen so far.
(2) Begin enumerating $W_{g(e,s)}$ by setting $W_{g(e,s)} = \emptyset$.
(3) Wait for a stage u such that $\Gamma(g(e,s), u) = 0$.
(4) Let $a \in A$ be such that $a = \Psi_e(b_e, s)$. If $a = 0$ do nothing. If $a \neq 0$, find integers c_m and c such that $ca = \sum_m c_m a_m$. Let $n(i) \in N$ be a fresh large number such that (1) the prime $p_{n(i)}$ does not occur in the decompositions of the coefficients c and c_m, (2) $h_{n(i),0} > 0$, and (3) $d(e)_{k,s} = 0$ for every $k \geqslant n(i)$.
(5) Enumerate $n(i)$ into $W_{g(e,s)}$. Keep $d(e)_{n(i),l} = 0$ for $l \geqslant s$ (unless we have a specific instruction not to do so). *Restrain* the element b_e by not allowing the construction to make it divisible by any prime greater than $p_{n(i)}$.
(6) Wait for one of the following three things to happen:
   I. $\Psi_e(b_e, s) \neq \Psi_e(b_e, t)$ at a later stage t. Then declare b_e not restrained and restart the strategy with t in place of s (go to (1); for instance, make b_e divisible by the corresponding power of $p_{n(i)}$).
   II. The number $n(i)$ enters the c.e. set $Inf(\alpha)$ at a stage $s > t$ (thus, $h_{n(i)} = \infty$). Make b_e infinitely divisible by $p_{n(i)}$ and return to (5) with $n(i+1)$ in place of $n(i)$ keeping b_e restrained.

III. $\Gamma(g(e,s),t)=1$ (thus, we believe $W_{g(e,s)} \cap Fin(\alpha) \neq \emptyset$ and $j \in Fin(\alpha)$). We remove the restraint from the element b_e allowing the construction to make b_e divisible by p_i with $i \notin W_{g(e,s)}$ if needed. We keep b_e not divisible by $p_{n(i)}$.
If at a later stage r the number $n(i)$ enters $Inf(\alpha)_r$ (thus, $W_{g(e,s),r} \subseteq Inf(\alpha)_r$), then make b_e infinitely divisible by $p_n(i)$. In this case also wait for a stage $w \geqslant r$ such that $\Gamma(g(e,s),w)=0$. Then return to (4) with a new fresh and large $n(j)$.

End of strategy.

Construction. At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively.

At stage s, let strategies R_e, $e \leqslant s$, act according to their instructions. If R_e acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make $\chi(a_n)=(h_{i,s})_{i\in\omega}$ in A_s for every $n \leqslant s$, and $(h_{i,s})_{i\in\omega}=(d(e)_{i,s})_{i\in\omega}$ in B_s for every $e \leqslant s$ which is not restrained, unless R_e keeps $d(e)_{i,s}=0$.
End of construction.

Verification. If $\lim_t \Psi_e(b_e,t)$ does not exist, then we reach I of (6) infinitely often and, therefore, $\chi(b_e)=\alpha$. Assume that $\lim_t \Psi_e(b_e,t)$ exists. Let s be the stage after which $\Psi_e(b_e,t)$ never changes again and

$$\Psi_e(b_e,s)=\lim_t \Psi_e(b_e,t).$$

Let $u \geqslant s$ be a stage such that $\lim_t \Gamma(g(e,s),t)=\Gamma(g(e,s),u)$.

The set $W_{g(e,s)}$ is designed to make $\lim_t \Gamma(g(e,s),t)=1$. If $\Gamma(g(e,s),u)=0$ was the case, then we would add more elements to $W_{g(e,s)}$ at a stage $v \geqslant u$ and eventually put some $n(j) \in Fin(\alpha)$ into $W_{g(e,s)}$, a contradiction.

By the definition of Γ, if $\lim_t \Gamma(g(e,s),t)=1$, then there is at least one $j \in W_{g(e,s)} \cap Fin(\alpha)$. Furthermore, the strategy guarantees that there is exactly one such a j, namely the last witness $n(i)$ which visits III of the strategy at some stage and stays there from this stage on. As a consequence, the element b_e will eventually be unrestrained (see the construction).

The algebraic strategy guarantees b_e is not divisible by $p_{n(i)}$ while the image is. Furthermore, b_e is declared not restrained as soon as we reach III with $n(i)$, meaning that the characteristic of b_e satisfies the property $d(e)_j=\alpha_j$ for each $j \neq n(i)$. It remains to apply Theorem 2.7. $\square$

We note that in the proposition above the algebraic strategy from Proposition 6.6 would not succeed. Theorem 6.1 is proved. $\square$

Corollary 6.8. *For a c.e. set P, the following are equivalent*:

(1) *G_P has a Σ_2^0 excellent $\widehat{P}$-basis*;
(2) *G_P has a Σ_2^0-basis as a free $Q^{(P)}$-module*;
(3) *G_P has a Π_1^0-basis as a free $Q^{(P)}$-module*;
(4) *G_P is Δ_2^0-categorical*;
(5) *$\widehat{P}$ is semi-low.*

Proof. The proof is a combination of Theorem 6.1, Theorem 5.5, and Lemma 3.8. $\square$

Corollary 6.9. *Each computable copy of the free abelian group of rank ω has a Π_1^0 set of free generators.*

Proof. The free abelian group can be viewed as the free Z-module. It remains to apply Theorem 5.5 and Theorem 6.1 with $\widehat{P}$ the set of all primes. $\square$

7. Concluding remarks and open questions

The notion of S-independence seems to be a natural generalization of linear independence to the case of free modules.

Problem 7.1. Study the effective content of S-independence and p-independence.

The effective content of p-independent sets (for a single prime p) seems to be unstudied. As we mentioned in the introduction, p-independent sets play an important role in the theory of primary abelian groups.

Problem 7.2. Generalize the results of the paper to non-homogeneous completely decomposable groups.

We expect that there is $n \in \omega$ such that every completely decomposable group is Δ_n^0-categorical. Can we describe all Δ_2^0-categorical completely decomposable groups? What is the complexity of the index set of all computable completely decomposable groups?

The theory of completely decomposable groups is an example of a beautiful and non-trivial mathematical theory having a number of pleasant results, especially in the countable case.

Problem 7.3. Study the reverse mathematics of completely decomposable abelian groups.

Limitwise monotonic sets were mentioned in the introduction. Recently the notion of a *limitwise monotonic sequence* proved to be useful in computable model theory [27]. Note that a c.e. characteristic can be viewed as a limitwise monotonic sequence in $(\omega \cup \{\omega\})^\omega$.

Problem 7.4. Study limitwise monotonic sequences in $(\omega \cup \{\omega\})^\omega$ having a computable settling time (see Definition 6.2). Do they have another applications in computable model theory?

We also expect that the results of the paper have analogs for modules over computably presentable principal ideal domains.

References

[1] B. Andersen, A. Kach, A. Melnikov, D. Solomon, Jump degrees of torsion-free abelian groups, J. Symbolic Logic, in press.
[2] C.J. Ash, Recursive labeling systems and stability of recursive structures in hyperarithmetical degrees, Trans. Amer. Math. Soc. 298 (1986) 497–514.
[3] C.J. Ash, J.F. Knight, Computable Structures and the Hyperarithmetical Hierarchy, Elsevier, Amsterdam, 2000.
[4] R. Baer, Abelian groups without elements of finite order, Duke Math. J. 3 (1937) 68–122.
[5] E.J. Barker, Back and forth relations for reduced abelian p-groups, Ann. Pure Appl. Logic 75 (1995) 223–249.
[6] B. Csima, D. Hirschfeldt, J. Knight, R. Soare, Bounding prime models, J. Symbolic Logic 69 (4) (2004) 1117–1142.
[7] J. Carson, V. Harizanov, J. Knight, K. Lange, C. McCoy, A. Morozov, S. Quinn, C. Safranski, J. Wallbaum, Describing free groups, Trans. Amer. Math. Soc. 364 (11) (2012) 5715–5728.
[8] M. Dehn, Über unendliche diskontinuierliche gruppen, Ann. of Math. 71 (1911) 116–144.
[9] R. Downey, On presentations of algebraic structures, in: Complexity, Logic, and Recursion Theory, in: Lect. Notes Pure Appl. Math., vol. 187, 1997, pp. 157–205.
[10] R. Downey, Computability theory and linear orderings, in: Handbook of Recursive Mathematics, Vol. 2, in: Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 823–976.
[11] R. Downey, C. Jockush, E. Schupp, Asymptotic density and computably enumerable sets, in press.
[12] R. Downey, S. Goncharov, A. Kach, J.F. Knight, O. Kudinov, A. Melnikov, D. Turetsky, Decidability and computability of certain torsion-free abelian groups, Notre Dame J. Form. Log. 51 (2010) 85–96.
[13] R.G. Downey, A.M. Kach, D. Turetsky, Limitwise monotonic functions and their applications (English summary), in: Proceedings of the 11th Asian Logic Conference, World Sci. Publ., Hackensack, NJ, 2012, pp. 59–85.
[14] R. Downey, A. Montalbán, The isomorphism problem for torsion-free abelian groups is analytic complete, J. Algebra 320 (2008) 2291–2300.
[15] Yu.L. Ershov, S.S. Goncharov, Constructive Models, Nauchnaya Kniga, Novosibirsk, 1999.
[16] E.B. Fokina, I. Kalimullin, R. Miller, Degrees of categoricity of computable structures, Arch. Math. Logic 49 (1) (2010) 51–67.

[17] Albrecht Fröhlich, John C. Shepherdson, Effective procedures in field theory, Philos. Trans. R. Soc. London Ser. A 248 (1956) 407–432.
[18] L. Fuchs, Infinite Abelian Groups, Vol. I, Academic Press, 1975.
[19] S. Goncharov, Countable Boolean Algebras and Decidability, Siberian School Algebra Logic, Nauchnaya Kniga, Novosibirsk, 1996.
[20] S. Goncharov, Autostability of models and abelian groups, Algebra Logic 19 (1980) 13–27 (English translation).
[21] Grete Hermann, Die Frage der endlich vielen Schritte in der Theorie der Polynomideale, Math. Ann. 95 (1) (1926) 736–788.
[22] G. Higman, Subgroups of finitely presented groups, Proc. R. Soc. London 262 (1961) 455–475.
[23] D. Hirschfeldt, R. Miller, S. Podzorov, Order-computable sets, Notre Dame J. Form. Log. 48 (3) (2007) 317–347.
[24] N. Khisamiev, Constructive abelian groups, in: Handbook of Recursive Mathematics, Vol. 2, in: Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 1177–1231.
[25] N. Khisamiev, On a class of strongly decomposable abelian groups, Algebra Logika 41 (4) (2002) 493–509, 511–512.
[26] B. Khoussainov, A. Nies, R. Shore, Computable models of theories with few models, Notre Dame J. Form. Log. 38 (2) (1997) 165–178.
[27] I. Kalimullin, B. Khoussainov, A. Melnikov, Limitwise monotonic sequences and degree spectra of structures, Proc. Amer. Math. Soc., in press.
[28] S. Lang, Algebra, Springer, 2002.
[29] P. LaRoche, Recursively presented Boolean algebras, Notices Amer. Math. Soc. 24 (1977) 552–553.
[30] C. McCoy, J. Wallbaum, Describing free groups, Part II: Π_4 hardness and no Σ_2 basis, Trans. Amer. Math. Soc. 364 (11) (2012) 5729–5734.
[31] W. Maass, Characterization of recursively enumerable sets with supersets effectively isomorphic to all recursively enumerable sets, Trans. Amer. Math. Soc. 279 (1983) 311–336.
[32] Anatolii I. Mal'cev, Constructive algebra I, Russian Math. Surveys 16 (1961) 77–129.
[33] A.I. Mal'tsev, On recursive abelian groups, translated in: Soviet Math. Dokl. 32 (1962) 1431–1434.
[34] C. McCoy, Categoricity in boolean algebras and linear orderings, Ann. Pure Appl. Logic 119 (2003) 85–120.
[35] A.G. Melnikov, $0''$-categorical completely decomposable torsion-free abelian groups, in: Proceedings of CiE 2009, in: Lecture Notes in Comput. Sci., vol. 5635, Springer, 2009, pp. 362–371.
[36] A.G. Melnikov, Computable ordered abelian groups and fields, in: Proceedings of CiE 2009, in: Lecture Notes in Comput. Sci., vol. 6158, Springer, 2010, pp. 321–330.
[37] A. Melnikov, Enumerations and completely decomposable torsion-free abelian groups, Theory Comput. Syst. 45 (4) (2009) 897–916.
[38] G. Metakides, A. Nerode, Recursively enumerable vector spaces, Ann. Pure Appl. Logic 11 (2) (1977) 147–171.
[39] A.T. Nurtazin, Computable Classes and Algebraic Criteria of Autostability, Summary of Scientific Schools, Math. Inst. SB USSRAS, Novosibirsk, 1974.
[40] M.O. Rabin, Computable algebra, general theory and theory of computable fields, Trans. Amer. Math. Soc. 95 (1960) 341–360.
[41] J.B. Remmel, Recursively categorical linear orderings, Proc. Amer. Math. Soc. 83 (1981) 387–391.
[42] R.I. Soare, Computational complexity, speedable and levelable sets, J. Symbolic Logic 42 (1977) 545–563.
[43] R.I. Soare, Recursively Enumerable Sets and Degrees, Springer-Verlag, Berlin Heidelberg, 1987.
[44] Bartel L. van der Waerden, Eine Bemerkung über die Unzerlegbarkeit von Polynomen, Math. Ann. 102 (1) (1930) 738–739.